# SYMMETRIC FORMULATION OF THE KALMAN-YAKUBOVICH-POPOV LEMMA AND ITS APPLICATION TO DISTRIBUTED CONTROL OF POSITIVE SYSTEMS

BY

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DISSERTATION

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# Abstract

This thesis makes three theoretical contributions to the robust system analysis and control theory. First, we revisit a central theorem in robust control theory known as the the Kalman-Yakubovich-Popov (KYP) lemma, and uncover its "hidden" symmetric structure that is rarely articulated in the literature of robustness analysis. Roughly speaking, we propose a new formulation of the KYP lemma so that two seemingly different quantities, "frequency" and "system uncertainties" play symmetric roles in the robust stability analysis. It turns out that the new formulation has sufficient generality to unify some of the recent extensions of the KYP lemma. Further consideration of this symmetry naturally leads us to the notion of mutual losslessness, which is the exact condition for the lossless of the analysis. As a result, the new formulation provides a general framework that answers when the KYP-like robustness analysis is lossless. Second, we restrict our focus to the class of cone-preserving linear dynamical systems. Square MIMO transfer functions in this class have what we call the DC-dominance property: the spectral radius of the transfer function attains its maximum at zero frequency and hence, the stability of the interconnected transfer functions is guaranteed solely by the static gain analysis. Using this property, we prove the delay-independent stability of cone-preserving delay differential equations. This provides an alternative proof of the delay-independent mean-square stability of multi-dimensional geometric Brownian motions. Finally, we further restrict our focus to the special class of cone-preserving systems known as positive systems. We prove a novel "diagonal" KYP lemma for positive systems, which ensures the existence of a diagonal storage function without introducing conservatism whenever the system is contractive. This result suggests that a certain class of distributed optimal control for positive systems can be found via the semidefinite programming (SDP).

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# Summary

The Kalman-Yakubovich-Popov (KYP) lemma is a central tool in the robust control theory. The KYP lemma converts various robust stability and performance conditions to Linear Matrix Inequality (LMI) conditions, so that they can be efficiently verified by semidefinite programming (SDP) solvers. Depending on the types of the analysis, the KYP lemma may provide a necessary and sufficient LMI condition to the desired system property (the KYP lemma is said to be lossless), but often the LMI condition is only sufficient (the KYP lemma is lossy). In this thesis, we formulate the KYP lemma in a general form so that it unifies some of the recent results on the KYP-based system analysis, with a particular emphasis on its underlying symmetric structure. Roughly speaking, this symmetry implies an intimate relationship between frequency variables and system uncertainties, storage functions and supply rate in the language of dissipativity theory, and also between parameter dependent Lyapunov functions and integral quadratic constraints (IQC). The main motivation for this work is to provide a unified theory to answer the question as to when the KYP lemma provides an exact LMI test for robust stability and performance analysis. The notion of mutual losslessness is introduced to characterize lossless S-procedures and the KYP lemma. It turns out that the new framework has sufficient flexibility to explain the losslessness of various robust stability analyses, including the Generalized KYP lemma for finite frequency analysis, the KYP lemma for nD systems, $\mu$-analysis, and the "diagonal" KYP lemma for positive systems. Although there is so far no general method for proving the mutual losslessness property, the proposed framework provides new intuitions on the existing and future robust control theories.

The complexity of the robust stability and performance analysis can be drastically reduced by encompassing special properties of the problem of interest. As the second main topic of this thesis, we focus on the class of linear dynamical systems that leaves a proper cone in the space of $L_2$ signals invariant, and uncover what we call the DC-dominant property of this class of systems. In short, this means that if $G(j\omega)$ is a cone-preserving square MIMO transfer function, then the spectral radius $\rho(G(j\omega))$ attains its maximum at $\omega = 0$. This property eliminates a need for the "frequency sweep" often required for the system analysis in generic theories and allows one to focus only on static gains of the system. Besides its direct implications, the DC-dominance property also contributes to simplify the analysis of the systems with delay. In particular, we prove that the delay-independent stability of cone-preserving

systems using this property. Using this result, we present an alternative approach to prove the delay-independent mean-square stability of multi-dimensional geometric Brownian motions.

We then focus on the special class of cone-preserving systems known as positive systems. Positive systems naturally appear to model real-world dynamical systems whose state variables are non-negative, such as temperature, probability, concentration of substances in chemical processes. We prove a novel "diagonal" KYP lemma for internally positive systems, which ensures the existence of a diagonal storage function without introducing conservatism whenever the system is contractive. This result suggests that a structured (or distributed) static state feedback $H_\infty$ optimal control design problem can be formulated as a convex problem and can be efficiently solved by a semidefinite programming (SDP) solver, provided the closed loop dynamics is again guaranteed to be internally positive. This makes the class of positive system interesting in the research of distributed control as well, since a convex formulation of the same control problem is not known for the general linear systems. Together with the LP-based control design already reported in the literature which is aiming at the $L_\infty$-induced gain minimization, now the convex formulation of the optimal distributed control in the $H_\infty$ sense is available for positive systems.

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# List of Symbols and Abbreviations

| | |
|---|---|
| $\mathbb{R}$ | Real numbers. |
| $\mathbb{Z}$ | Integers. |
| $\mathbb{N}$ | Natural numbers (Positive integers). |
| $\mathbb{C}$ | Complex numbers. |
| $\mathbb{C}_+$ | Open right half plane. |
| $\bar{\mathbb{C}}_+$ | Closed right half plane. |
| $\bar{D}$ | Closed unit disc. |
| $\mathbb{H}_n$ | $n \times n$ Hermitian matrices. |

| | |
|---|---|
| BMI | Bilinear Matrix Inequality |
| FDI | Frequency Domain Inequalities |
| KYP | Kalman-Yakubovich-Popov (Lemma) |
| IQC | Integral Quadratic Constraint |
| LFT | Linear Fractional Transformation |
| LMI | Linear Matrix Inequality |
| LP | Linear Programming |
| LTI | Linear Time Invariant |
| MIMO | Multi-Input Multi-Output |
| SDP | Semidefinite Programing |
| SISO | Single-Input Single-Output |

# Chapter 1

# Introduction

## 1.1 Overview

It is virtually impossible to obtain a perfect mathematical model for a physical, social, engineering and cyber system in the real world. Nevertheless, we are often required to confirm the safety of a system based on its imperfect information, either in a deterministic or probabilistic fashion, in order to operate them properly. For the system and control theory to be a versatile player in these ubiquitous needs, it is important to have a rich collection of mathematical methods to describe various forms of uncertainties. Once the uncertainties are appropriately expressed as a mathematical model, it is also important to have an efficient algorithm to analyze the constructed model and verify its robustness. Throughout this thesis, such a two-step approach to analyze the real world systems is referred to as the *robustness analysis*. These two requirements in the robustness analysis are often trade-off: if one wish to use a very precise and detailed description of the uncertainty, there is typically no efficient algorithm to analyze it, while easily analyzable uncertainty models likely to result in a conservative result. Moreover, definitions of the "richness" of the collection of uncertainty descriptions and the "efficiency" of the algorithms evolves over the decades. For example, a previously believed "rich enough" collection of robustness analysis tools can turn out to be mostly fragile in emerging applications [1]. Similarly, many of the LMI-based robustness analysis were not considered efficient until sophisticated SDP solvers became available.

In this introductory chapter, we review a method to describe uncertainties using the well-posedness model, also known as the LFT (linear fractional transformation) model or the $M-\Delta$ model, which is reasonably flexible to accommodate many types of robustness analysis and is reasonably simple to analyze. (In this sense, the well-posedness model locates on the Pareto frontier of the bidirectional requirements in the robustness analysis as in Figure 1.1.) Among other possible definitions appearing in system and control literature, our definition of the well-posedness is particularly simple.

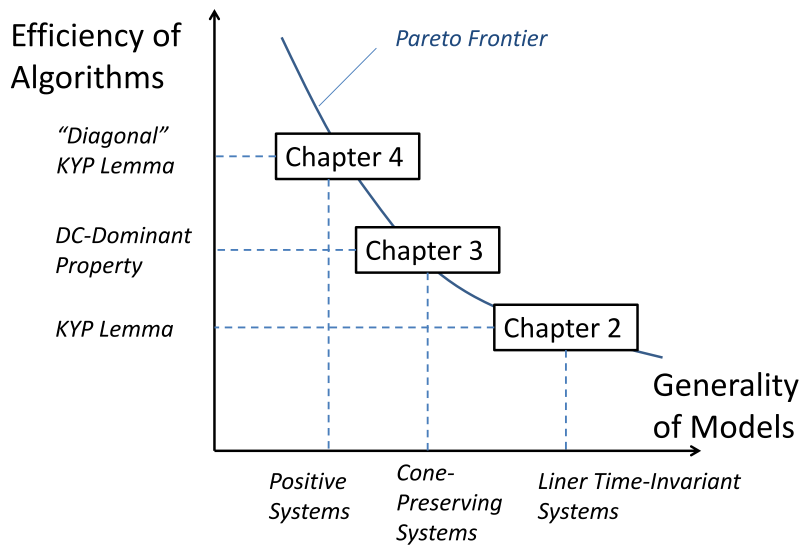


Figure 1.1: Overview of this thesis

We restrict our attention to the feedback loop of interconnected matrices (not operators), and concerns invertibility of the resolvent matrix over a certain domain. An implicit but an important assertion of this thesis is that this framework suffices for the robust stability analysis of linear time invariant (LTI) systems. This observation is related to the symmetric formulation of the KYP lemma in Chapter 3, in which a frequency variable (such as the Laplace variable "$s$") is treated as an unknown complex number, as in the same way as other uncertain parameters in the dynamical system are treated. In the following sections, we will discuss how this a robust stability analysis problem can be formulated as a well-posedness problem.

When a robustness analysis is performed, it is important to exploit a special "structure" of the system of interest, since much more efficient algorithms than a general purpose algorithm might be available by making use of the structure. In Chapter 3 and 4, we proceed the path along the Pareto frontier as in Figure 1.1, and consider a special class of systems with a particular structure and more efficient algorithms for exploiting the structure. In Chapter 3, we will focus on a special class of linear systems that has the "cone-preserving" property. Dynamical systems in this class appear in statistics, signal processing and filtering algorithms, as well as in quantum systems. Also, the entire class of positive systems belongs to this class. We will make use of the special structure of the systems in this class and prove that it is sufficient to perform a static gain (DC-gain) analysis, i.e., the behavior of the system at zero frequency to conclude stability. This gives a drastically simpler criterion for the well-posedness as compared to the scenario without taking cone-preserving property into account. Then we further restrict out attention to positive systems in Chapter 4.

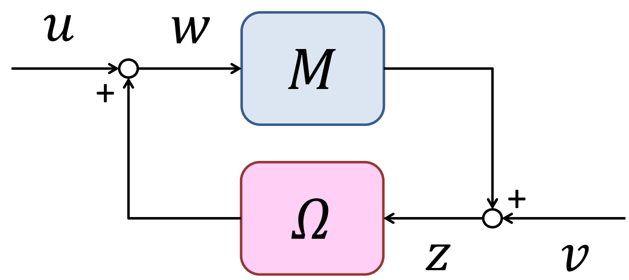


Figure 1.2: Interconnected matrices

## 1.2 Well-posedness of an algebraic loop

Since the notion of well-posedness used in this thesis is very simple, we start by introducing its mathematical definition. Suppose two matrices $M, \Omega \in \mathbb{C}^{n\times n}$ are given, and vectors $u, v, x, y \in \mathbb{C}^n$ satisfy the following set of equations:

$$z = Mw + v, \quad w = \Omega z + u. \tag{1.1}$$

This relationship is graphically written as in Figure 1.2. We assume that $M$ is a fixed matrix, while $\Omega$ takes values in a set $\mathbf{\Omega} \subset \mathbb{C}^{n\times n}$.

**Definition 1** *(Well-posedness) Let a matrix $M \in \mathbb{C}^{n\times n}$ and a non-empty set $\mathbf{\Omega} \subseteq \mathbb{C}^{n\times n}$ be given. The* interconnection *of $M$ and $\mathbf{\Omega}$, denoted by $[M, \mathbf{\Omega}]$, is said to be well-posed if*

$$\begin{bmatrix} I & -M \\ -\Omega & I \end{bmatrix}^{-1} \textit{exists for all } \Omega \in \mathbf{\Omega}, \textit{ and there is } \gamma > 0 \textit{ such that } \left\| \begin{bmatrix} I & -M \\ -\Omega & I \end{bmatrix}^{-1} \right\| \leq \gamma \ \forall \Omega \in \mathbf{\Omega}.$$

In other words, the well-posedness of $[M, \mathbf{\Omega}]$ means that the induced gain of a linear map $f_\Omega : (u, v) \mapsto (z, w)$ is uniformly bounded over $\mathbf{\Omega}$. Well-posedness in this definition is arguably simpler than the widely accepted definition involving operators on Hilbert spaces (e.g. [22]). Although we will keep using the terminology *well-posedness*, the condition in Definition 1 might be better referred to as the *robust matrix invertibility*. It is somewhat surprising to observe that many engineering problems can be reduced to the robust matrix invertibility problems.

## 1.3 Modeling dynamics via well-posed feedback loop

Suppose the behavior of a dynamical system is modeled as a differential equation $\dot{x}(t) = Ax(t)$ and there is no uncertainty. Then the system is asymptotically stable if and only if the matrix $A$ is Hurwitz, i.e., there is no eigenvalues in the closed right half plane. The fact that this system is asymptotically stable can be equivalently written as a well-posedness condition.

### 1.3.1 Example : Continuous time linear dynamical system

A linear differential equation $\dot{x}(t) = Ax(t)$ is asymptotically stable if and only if the interconnection $[A, \mathbf{\Lambda}]$ is well-posed where

$$\mathbf{\Lambda} = \{\lambda I : \lambda \in \bar{\mathbb{C}}_+\}.$$

To see the sufficiency, suppose that the transfer function is not stable, i.e., $A$ has an eigenvalue in $\bar{\mathbb{C}}_+$. If $A$ has an eigenvalue at the origin, the interconnection cannot be well-posed for the following reason. Consider a sequence $\{\lambda_k\}$ in $\bar{\mathbb{C}}_+ \setminus \{0\}$ such that $\lim_{k\to\infty} \lambda_k = \infty$. Then

$$\lim_{k\to\infty} \|(I - \lambda_k A)^{-1}\lambda_k\| = \lim_{k\to\infty} \|(\frac{1}{\lambda_k} I - A)^{-1}\| = \infty.$$

By the matrix inversion lemma,

$$\lim_{k\to\infty} \left\| \begin{bmatrix} I & -A \\ -\lambda_k I & I \end{bmatrix}^{-1} \right\| = \lim_{k\to\infty} \left\| \begin{bmatrix} I + A(I-\lambda_k A)^{-1}\lambda_k & A(I-\lambda_k A)^{-1} \\ (I-\lambda_k A)^{-1}\lambda_k & (I-\lambda_k A)^{-1} \end{bmatrix} \right\| = \infty \tag{1.2}$$

The above limit is unbounded because the lower left corner is unbounded. This contradicts the well-posedness. On the other hand, if $A$ has an eigenvalue $\nu$ in $\bar{\mathbb{C}}_+ \setminus \{0\}$, by choosing $\lambda = \nu^{-1} \in \bar{\mathbb{C}}_+$, $\lambda A$ has an eigenvalue at $1$. This means that $\begin{bmatrix} I & -A \\ -\lambda I & I \end{bmatrix}$ is not invertible, and again contradicts the well-posedness. Hence, the well-posedness of $[A, \Lambda]$ is sufficient for stability.

To see the necessity, suppose $[A, \mathbf{\Lambda}]$ is not well-posed. Then there exists a sequence $\{\lambda_k\}$ in $\bar{\mathbb{C}}_+ \setminus \{0\}$ such that (1.2) holds and

$$\lim_{k\to\infty} \lambda_k = \lambda_\infty \in \bar{\mathbb{C}}_+ \cup \{\infty\} \setminus \{0\}.$$

Here, we have already removed the possibility that $\lambda_\infty = 0$, since in this case (1.2) cannot hold. So the sequence

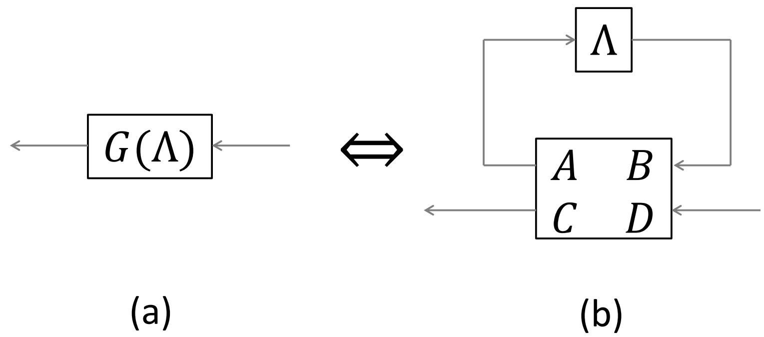


Figure 1.3: LFT representation of a transfer function

$\nu_k := 1/\lambda_k$ has a bounded limit $\lim_{k\to\infty} \nu_k = \nu_\infty = 1/\lambda_\infty \in \bar{\mathbb{C}}$, and

$$(1.2) = \lim_{k\to\infty} \left\| \begin{bmatrix} I + A(\nu_k I - A)^{-1} & A(\nu_k I - A)^{-1}\nu_k \\ (\nu_k I - A)^{-1} & (\nu_k I - A)^{-1}\nu_k \end{bmatrix} \right\| = \infty.$$

For this to be true, it is necessary that

$$\lim_{k\to\infty} \|(\nu_k I - A)^{-1}\| = \infty.$$

This implies that $A$ has an eigenvalue at $\nu_\infty \in \bar{\mathbb{C}}_+$. Hence $\dot{x}(t) = Ax(t)$ cannot be asymptotic stable.

### 1.3.2 Example : Discrete time linear dynamical system

A linear difference equation $x(k+1) = Ax(k)$ is asymptotically stable if and only if the interconnection $[A, \mathbf{\Lambda}]$ is well-posed where

$$\mathbf{\Lambda} = \{\lambda I : \lambda \in \bar{D}\}.$$

This can be verified similarly as in the previous example.

These observations indicate that a stable transfer function $\hat{G}(s) = C(sI - A)^{-1}B + D$ can be written using a well-posed algebraic loop as in Figure 1.3. In this figure, the matrix $\Lambda$ can take values in a user specified domain $\mathbf{\Lambda}$ (two examples are considered above), and is understood be the classical integrator blocks $s^{-1}I$ or $z^{-1}I$. We may refer to a matrix values quantity $\Lambda$ itself as the frequency variable, and write the same transfer function as $G(\Lambda) = C(I - \Lambda A)^{-1}\Lambda B + D$. As we will see later, the symmetric role between frequency variables and system uncertainties can be better seen through $\Lambda$ than through $s$ or $z$.

## 1.4 Modeling uncertainties via well-posed feedback loop

When there is no uncertainty in the model $\dot{x}(t) = Ax(t)$, the stability can be checked by constructing the characteristic polynomial of $A$ and apply the Routh-Hurwitz criterion. Alternatively, one can search for a positive definite matrix $P$ satisfying the Lyapunov inequality $A^T P + PA < 0$ by semidefinite programming. Both algorithms are known to be efficient, in the sense that the time complexity grows at most as a polynomial function of the size of the problem instance [2].

If the linear equation model $\dot{x}(t) = Ax(t)$ contains uncertainties, there are many possible approaches to guarantee robust stability. One natural approach inspired by the Routh-Hurwitz criterion is to capture the uncertainty by interval polynomials. This approach assumes that coefficients of the characteristic polynomials take values in some prespecified interval and asks if all polynomials in this set are Hurwitz. A celebrated result in this approach is the Kharitonov's theorem [3], which states that the robust stability is guaranteed if and only if four important corner polynomials are Hurwitz. This surprising result is valuable in the computational aspect as well, since it implies that the time complexity of the algorithm grows at most polynomially as the order of the system grows. One criticism to this approach is that the interval polynomial model is often unrealistic in practice, since in many cases, the uncertain coefficients are correlated to each other and not "fully perturbed" in the uncertainty set. This issue is partially solved by using a polytope of polynomials rather than interval polynomials and applying the so-called the Edge theorem. Alternatively, one can regard the entries of the matrix $A$ are uncertain. Unfortunately, there is no similar result to the Kharitonov's theorem for a matrix interval. In fact, it is known that robust stability of matrix interval is NP-hard to decide.

It is also standard to consider the small gain uncertainties of the form of $\dot{x} = (A + B\Delta C)x$, in which that the unknown (complex) matrix $\Delta$ is assumed to satisfy $\|\Delta\| \leq \rho$. The largest $\rho$ to which the asymptotic stability is guaranteed is called the (complex) stability radius. The same quantity when $\Delta$ is restricted to be a real matrix is called the real stability radius. The real stability radius is known to be much harder to compute than the complex stability radius, but an explicit formula is derived in [4]. In these models, $\Delta$ is assumed to be completely deterministic, but the other extreme (in the sense that $\Delta$ is a stochastic perturbation) is also popular model of uncertainties. The largest $\rho$ to which the stochastic differential equation $dx = Axdt + \rho Bxdw$ is mean-square stable is also called the stability radius, and is extensively studied (e.g., [5, 6, 7]).

Much more general form of deterministic uncertainties can be modeled by using the *linear fractional transformation* (LFT) by $\dot{x} = A(\Delta)x$, $A(\Delta) = A + L(I - \Delta D)^{-1}\Delta R$, where $\Delta$ belongs to a certain prescribed set $\boldsymbol{\Delta}$. This form is known to be general enough to model uncertain parameters entering $A(\Delta)$ in a polynomial or rational manner, provided that $A(\Delta)$ is not singular over $\boldsymbol{\Delta}$. This is known as the LFT representation lemma [8, 9]. To show the

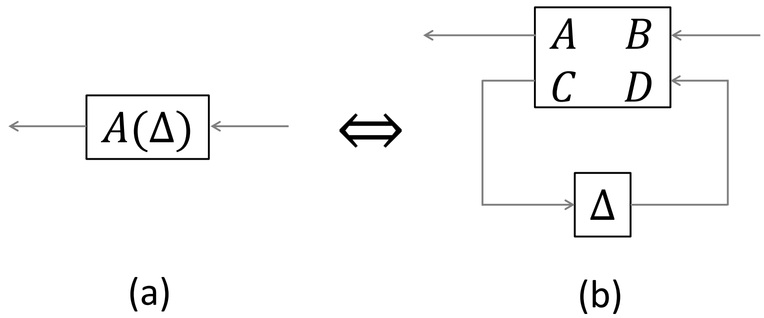


Figure 1.4: LFT representation of an uncertainty

flexibility of the LFT representation using an example, consider the following uncertain Vandermonde matrix:

$$A(\delta) = \begin{bmatrix} 1 & x_1 + \delta_1 & \cdots & (x_1 + \delta_1)^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_m + \delta_m & \cdots & (x_m + \delta_m)^{n-1} \end{bmatrix}$$

where $\delta_i$'s are unknown but bounded parameters. This can be written in the form of $A(\delta) = A + B(I - \Delta D)^{-1}\Delta C$ where matrices $A, B, C, D$ are "certain" quantities while uncertain quantities $\delta_i$ appear on the diagonals of a diagonal matrix $\Delta$, provided that $(I - \Delta D)^{-1}$ exists [10]. This formulation is used for robustness analysis of polynomial interpolation. Notice that the uncertainties expressed in the LFT model can be graphically written as in Figure 1.4.

## 1.5 Robust stability analysis using well-posedness model

We have seen that the stability of a transfer function is equivalent to the well-posedness of an algebraic loop in Figure 1.3. We also saw that uncertainties that enters into a matrix $A$ in a polynomial or a rational way can be written as an algebraic loop for a specific $\Delta$ in Figure 1.4. By combining these observations, the robust stability of the system $\dot{x} = A(\Delta)x$ can be equivalently written as the well-posedness condition of the interconnection $[M, \mathbf{\Omega}]$, where

$$M = \begin{bmatrix} A & L \\ R & D \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{\Lambda} & 0 \\ 0 & \mathbf{\Delta} \end{bmatrix}.$$

In the above expression, the set $\mathbf{\Lambda}$ is typically a user specified frequency region, while the set $\mathbf{\Delta}$ is a user specified uncertainty region. In a block diagram form, the robust stability can be expressed as the well-posedness of an algebraic loop in Figure 1.5, which can be obtained by combining Figure 1.3 and Figure 1.4. This operation can be thought of as a generalization of the main loop theorem [11]. In this formulation, the algebraic roles of the two matrices $\Lambda$ and $\Delta$ are

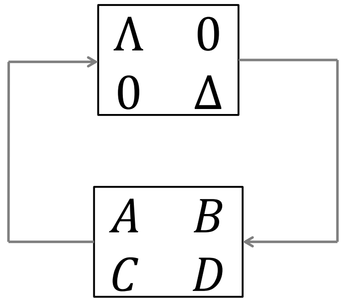


Figure 1.5: Generalized main loop theorem

symmetric, even though the physical meanings of these two objects are originally different. This observation suggests that all known techniques to characterize system uncertainties can also be used to characterize frequency regions, and vice versa.

## 1.6 Discussion

Our treatment is slightly unconventional in that we are treating the "frequency variable" $s$ and the uncertainty $\Delta$ equally. One might ask what if the uncertainty is not a simple matrix but a unknown transfer function $\Delta(s) = C_\Delta(sI - A_\Delta)^{-1}B_\Delta + D_\Delta$. This is very likely to happen in practice if one tries to capture unmodeled dynamics as uncertainties. One justification for us to focus on the matrix-valued $\Delta$ only is that we can always rewrite the feedback diagram in such a way that a nominal matrix $M$ is interconnected to uncertain *matrices* $A_\Delta, B_\Delta, C_\Delta, D_\Delta$, as well as to another uncertain matrix $\Lambda = s^{-1}I$ (in fact, this is an integrator). Another justification is that the small gain theorem guarantees that if the interconnection of the nominal system $M(s)$ and a stable transfer function $\Delta(s)$ satisfying $\|\Delta(s)\|_\infty < 1$ is internally stable if and only if the interconnection of $M(s)$ and a *complex matrix* $\Delta$ satisfying $\|\Delta\| < 1$ is internally stable (e.g.,[12] p.219).

Our approach to the robustness analysis is based on the well-posedness condition defined in Definition 1, which simply concerns the invertibility of a resolvent *matrix*. More generally, the invertibility of resolvent operators in a Hilbert space can be employed as the definition of the well-posedness as in [13]. We take the former approach to pursue simplicity of the unified theory, but there is a cost of simplicity. One big restriction in our framework is that the treatment of time has to be through the frequency domain. This clearly means that our framework requires a dynamical system of interest to be expressed as a "transfer function" form through the Laplace transform, the $z$-transform, and etc. For example, one of the important results in the robust stability analysis is the necessity of the scaled small gain test against norm-bounded structured linear *time-variant* uncertainties (this is based on the lossless S-procedure on $L_2$ space by [14], a detailed explanation can be found in [13], Chapter 8). However, this result is difficult to recover in our framework, since the class of time-variant uncertainties is difficult to express in the frequency domain.

## 1.7 How to ensure well-posedness?

So far, we have seen that the well-posedness model provides a fairly general framework for robustness analysis. This observation itself, however, is of little of use in practice without an efficient algorithm to check if a given interconnection $[M, \mathbf{\Omega}]$ is well-posed or not. Since the set $\mathbf{\Omega}$ is typically uncountable, it is virtually impossible to verify the definition of well-posedness directly. Hence the existence of an algorithm to give a "certificate" for the well-posedness is crucial.

Unfortunately, verifying well-posendness can be computationally challenging even for a relatively simple and practically important structure of $\mathbf{\Omega}$. For example, suppose that the set $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = \{diag(\delta_1 I, \cdots, \delta_n I) : \ -1 \le \delta_i \le 1\}.$$

Then verifying the well-posedness of $[M, \mathbf{\Omega}]$ is essentially the (real) structured singular value problem ($\mu$-analysis). In [15], it is shown that computing $\mu$ is NP-hard by showing that the indefinite quadratic programming, which is already known to be NP-complete, can be cast as a problem of $\mu$ computation. There are several possible approaches to deal with the NP-hardness of the $\mu$ problem. For example, since $\mathbf{\Omega}$ is compact, the well-posedness is equivalently verified by showing $\det(I - M\Omega) \neq 0$ for all $\Omega \in \mathbf{\Omega}$. This means that the following closed semialgebraic set is empty.

$$\mathcal{C} = \left\{ (\delta_1, \cdots, \delta_n) \in \mathbb{R} \times \cdots \times \mathbb{R} : \begin{array}{l} f(\delta_1, \cdots, \delta_n) := \det(I - M\Omega) = 0 \\ \Omega = diag(\delta_1 I, \cdots, \delta_n I) \\ g_i(z) = 1 - \delta_i^2 \ge 0 \ \forall i = 1, \cdots, n \end{array} \right\}$$

From the viewpoint of real algebraic geometry, the Stingel's positivstellensatz [16, 17] implies that $\mathcal{C}$ is empty if and only if

$$-1 \in ideal(f) + cone(g_1, \cdots, g_n). \tag{1.3}$$

This means that there exists $k \in \mathbb{N}$ such that one can construct a particular set of sum of square (SOS) polynomials of order at most $k$ which guarantees the emptiness of $\mathcal{C}$ if it is empty. Since polynomial SOS problem can be written as a semidefinite programming problem, at least in principle, the $\mu$ problem is arbitrarily approached by a hierarchy of SDPs as far as one is allowed to keep increasing the dimension of the SDPs. Related techniques via the problem of moments are also known (e.g., [18, 19]). A drawback of this generic approach is that the dimension of the SDP grows quickly by increasing the order of polynomials, and the largest order of the polynomial required to check the above condition is not known *a priori*.

Another approach is to use polynomial time randomized algorithms [8] to assert that the set $\mathcal{C}$ is empty "with high probability". One advantage of this approach is its flexibility to encompass the probability distribution of the uncertainties and accept a small probability of violation, which practically makes sense in many real applications. Note that such randomized approach is effective not only for the well-posedness based robustness analysis, but also for many other engineering problems. However, the time complexity of the randomized algorithms required to generate sufficient number of samples grows dramatically as the size and the VC-dimension of the problem grows.

In order to circumvent time complexity issues of these generic approaches, it is often valuable to exploit a particular structure of the problem. As noticed in the literature of the $\mu$-analysis, the uncertain matrix $\Omega$ typically has a block diagonal structure as in Figure 1.6. The Kalman-Yakubovich-Popov (KYP) Lemma in our formulation in the following chapters can be viewed as a systematic method to relax the well-posedness problem of the form of Figure 1.6 to a conic programming problem (typically LMIs) to which an efficient numerical solvers are often available. As one can infer from the NP-hardness of the $\mu$-problem, the relaxation by the KYP lemma is not always tight. When the relaxation is not tight, an LMI condition deduced from the KYP lemma is only a sufficient condition for the well-posedness. In such cases, the KYP lemma is said to be *lossy*. On the other hand, if the LMI condition deduced from the KYP lemma is indeed tight, then the lemma is said to be *lossless*. In the next chapter, we consider the losslessness property of the KYP lemma in depth.

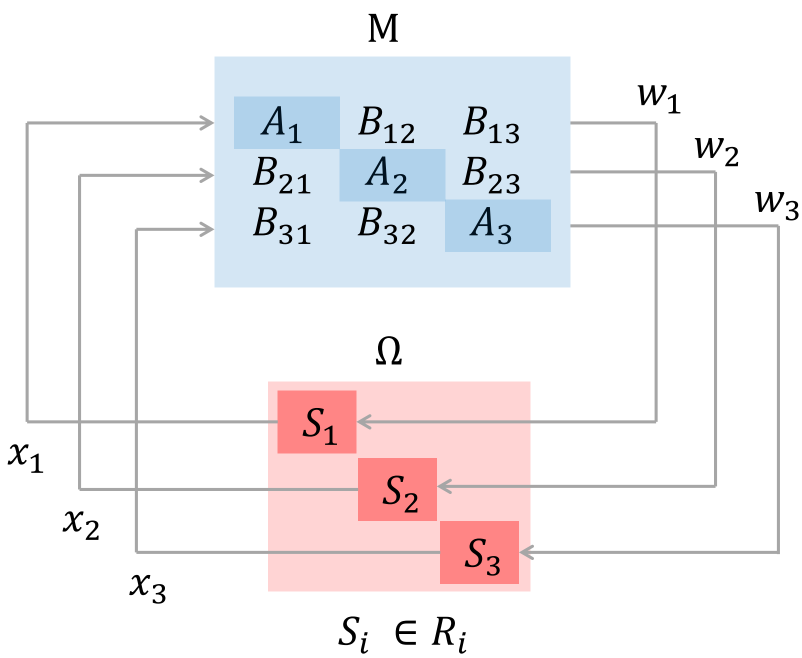


Figure 1.6: Well-posedness with block diagonal $\Omega$.

# Chapter 2

# Symmetry in the KYP Lemma and Its Losslessness

In this chapter, we consider algorithms to verify stability and performance of dynamical systems. Since we saw that the well-posedness problem is the "standard problem" which many robust analysis problems can be converted to, we are interested in algorithms that can efficiently prove well-posedness. We will focus on the well-posedness problem with a certain structure as shown in Figure 1.6 in which each of the matrix valued parameters $S_i$ belongs to some prespecified sets $\mathcal{R}_i$. Clearly, the difficulty of the analysis depends on the "shapes" of the sets $\mathcal{R}_i$. We want to have a rich and flexible framework to approximate various shapes of uncertainty sets so that the framework can accommodate various types of uncertainties appearing in practices. At the same time, we want to keep the shape of $\mathcal{R}_i$ simple so that efficient analysis algorithms exist to analyze them. A good compromise is to focus on the class of $\mathcal{R}_i$ that can be characterized by a set of quadratic forms (the definition will be given shortly). This leads to system analysis via the Kalman-Yakubovich-Popov (KYP) lemma, which can be seen as a unification of the historical techniques such as the small gain theorem, passivity theorem, Popov criterion, and the circle criterion. We regard the KYP lemma as a systematic method to convert well-posedness condition into a conic programming condition such as LMIs which is often verifiable by a polynomial time algorithm. For some problems, the conic programming yields a solution (a certificate) if and only if the system is well-posed (the KYP lemma is *lossless*). Unfortunately, for many important problems, such a certificate is only sufficient (the KYP lemma is *lossy*). Although various types of KYP-like lemmas are studied in the literature, their losslessness results appear rather sporadically and no transparent discussion has been made as to when the lossless KYP lemma is available. Hence in this chapter, we attempt to give a common theoretical framework that gives a big picture as to when the losslessness is available, by first noticing a symmetry between the multipliers used to characterize the uncertainties of interest, and the Lyapunov-like functions used to establish stability. The form of the KYP lemma we propose in this chapter has sufficient flexibility to express various types of the KYP-like lemmas and explain their losslessness.

## 2.1 The first example

The symmetry we will emphasize in this chapter can be clearly seen in the following example of discrete time linear model

$$G: \begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{array} .$$

For the sake of simplicity, we assume $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ here. Suppose that this systems is dissipative (in the sense of [20]) with respect to the supply rate $s(u, y)$ with storage function $V(x)$ given by

$$s(u, y) = y^T Qy - u^T Qu \; ; \; V(x) = x^T Px$$

where $P$ and $Q$ are positive definite matrices. The necessary and sufficient condition for this to be true is the matrix inequality condition

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \tag{2.1}$$

$$\Theta = \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}, \; \Pi = \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}.$$

Now, notice an apparent symmetry holding between $\Theta$ and $\Pi$ in the above condition. Namely, if we define a new system

$$\tilde{G}: \begin{array}{l} x(k+1) = Dx(k) + Cu(k) \\ y(k) = Bx(k) + Au(k), \end{array}$$

then the above condition is equivalent to saying that $\tilde{G}$ is dissipative with respect to the supply rate $\tilde{s}(u, y)$ with a storage function $\tilde{V}(x)$ given by

$$\tilde{s}(u, y) = y^T Py - u^T Pu \; ; \; \tilde{V}(x) = x^T Qx.$$

Notice that in the new interpretation, the roles of $P$ and $Q$ in the supply rate and the storage function are flipped. Among the voluminous literature related to the KYP lemma, this symmetry seems to be implied by a few but not many papers. The most explicit reference is given by [11], which points out the "duality" between the $H_\infty$ norm conditions and the parameter dependent Lyapunov conditions in the context of well-posedness analysis of uncertain LTI systems. Another indicator of this symmetry in the literature is the fact that the notion of S-procedure is used for both purposes of frequency domain specification and uncertainty specification. For example, the Generalized KYP lemma [21] utilizes a sophisticated lossless S-procedure to derive a tractable algorithm for a nontrivial frequency domain test, while the role of the S-procedure in the IQC framework is well known [22]. However, to the best of our knowledge, there is no literature discussing the KYP lemma with an explicit emphasis on this symmetry. With an emphasis on symmetry, we introduce a new notion of *mutual losslessness* to describe when the KYP lemma yields a lossless LMI condition [23]. It turns out that noticing this symmetry in the KYP lemma, together with the notion of mutual losslessness, gives a good intuition on the losslessness of various recently proposed robust stability analysis tools, including the Generalized KYP lemma [21], KYP lemma for nD-systems [24][25], and $\mu$-analysis.

## 2.2 Contribution of this chapter

It is not our intension to make this chapter a restatement or a review of the KYP lemma. Our contribution in this chapter lies in its formulation, so that the underlying mathematical structure is articulated. In particular, we emphasize the fact that both frequency variables (such as Laplace variable "$s$") and system uncertainties $\Delta$ can be modeled as one of uncertain parameters $S_i$ in Figure 1.6. Once the problem is formulated in this form, it becomes less important to know the original physical meaning of each uncertain parameter. Hence, in the LTI system analysis it is more natural to treat them symmetrically, although overwhelming literatures treat "frequency" and the "uncertainty" as different objects.

One benefit of highlighting the symmetry is that it clarifies the point that the losslessness of the KYP lemma should be naturally discussed as a matter of the relationship among sets of Hermitian matrices involved in the lemma. In this way, we propose a new notion, which we call the *mutual losslessness* to characterize the exact condition for losslessness. The notion of mutual losslessness can be seen as a generalization of the losslessness in the classical literatures on the S-procedure to the situations in which there are infinite number of constraints. This will be discussed in Section 2.9.

Although the new algebraic condition of mutual losslessness remains difficult to check, we believe that this observation opens new perspectives for understanding and proving losslessness of the S-procedure and the KYP lemma under various situations. In Section 2.7, we relate seemingly different questions in system analysis by switching the role of frequency variables and the system uncertainties. This suggests that the knowledge obtained for one problem can be directly applied to the other problem.

In short, the contributions of this chapter are: (1) A framework to characterize matrix valued regions both for frequency variables and uncertainties, which enables a unified description of various types of the KYP lemma, (2) Introduction of the symmetric formulation of the S-procedure and the KYP lemma, which has its own beauty and novelty, (3) The notion of mutual losslessness, which is a generalized way to characterize the lossless properties of the S-procedures.

## 2.3 Quadratic characterization

We first define a flexible framework to describe various types of regions $\mathcal{R}_i \subset \mathbb{C}^n$ for uncertain parameters $S_i$. An important observation is that the same framework is used to characterize both domains of frequency variables and system uncertainties. This allows us to describe the KYP lemma in a symmetric form in later sections. Let $\mathbf{\Theta} \subset \mathbb{H}_{2n}$ be a convex cone. Suppose the domain $\mathcal{R}_i$ to which parameter $S_i$ in Figure 1.6 belongs is characterized by $\mathbf{\Theta}$, and is given by

$$\mathcal{R}(\mathbf{\Theta}) := \left\{ S \in \mathbb{C}^{n\times n} : \begin{bmatrix} I \\ S \end{bmatrix}^* \Theta \begin{bmatrix} I \\ S \end{bmatrix} \geq 0 \quad \forall \Theta \in \mathbf{\Theta} \right\}. \tag{2.2}$$

The definition (2.2) can be used to specify a varieties of regions in $\mathbb{C}^{n\times n}$ by appropriately choosing the convex cone $\mathbf{\Theta} \subset \mathbb{H}_{2n}$. For example:

**Example 1** *If* $\mathbf{\Theta} = \left\{ \Theta \in \mathbb{H}_{2n} : \Theta = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix},\ P > 0 \right\}$, *then* $\mathcal{R}(\mathbf{\Theta}) = \left\{ S = sI : s \in \bar{\mathbb{C}}_+ \right\}$.

**Proof:** It is easy to show that $\mathcal{R}(\mathbf{\Theta}) \supset \left\{ S = sI : s \in \bar{\mathbb{C}}_+ \right\}$. To see the converse, suppose $S \in \mathcal{R}(\mathbf{\Theta})$. Pick $P = E_i + \epsilon I > 0$ where $E_i$ has all zero entries except its $(i,i)$-th entry, which is one. We obtain

$$\begin{bmatrix} I \\ S \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} 0 & * & 0 \\ * & 2Re(S_{ii}) & * \\ 0 & * & 0 \end{bmatrix} + \epsilon(S + S^*).$$

In order for $S \in \mathcal{R}(\mathbf{\Theta})$, the above matrix needs to be positive definite for arbitrarily small $\epsilon > 0$. To this to be true, it is necessary that $Re(S_{ii}) \geq 0$. Also, entries indicated by $*$ must be all zero. This shows that an element of $\mathcal{R}(\mathbf{\Theta})$ has to have a form $S = diag(s_1, \cdots, s_n)$ with $Re(s_i) \geq 0$ for all $i = 1, \cdots, n$. To show that $s_1 = \cdots = s_n$, suppose that $s_1 \neq s_2$ without loss of generality. By taking

$$P = \left[\begin{array}{cc|c} 1 & 1 & \\ 1 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right] + \epsilon I > 0,$$

we have

$$\begin{bmatrix} I \\ S \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \left[\begin{array}{cc|c} 2(1+\epsilon)Re(s_1) & s_2 + \bar{s}_1 & \\ s_1 + \bar{s}_2 & 2(1+\epsilon)Re(s_2) & 0 \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right]. \tag{2.3}$$

| | $\boldsymbol{\Theta}$ | Frequency region $\mathcal{R}(\boldsymbol{\Theta})$ | Applications |
|---|---|---|---|
| (a) | $\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} : P > 0$ | $\Lambda = \lambda I : \lambda \in \bar{\mathbb{C}}_+$ | Frequency domain for continuous time stable systems |
| (b) | $\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} : P > 0$ is diagonal | $\Lambda = diag(\lambda_1, \cdots, \lambda_n) :$ $\lambda_i \in \bar{\mathbb{C}}_+ \ \forall i = 1, \cdots, n$ | Frequency domain for stable n-D systems |
| (c) | $\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} : Q > 0$ | $\Lambda = \lambda I, \lambda \in \bar{\mathbb{D}}$ | Frequency domain for discrete time stable systems |
| (d) | $\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} :$ $Q = diag(Q_h, Q_v)$ $Q_h \in \mathbb{H}_{n_h}, Q_v \in \mathbb{H}_{n_v}$ | $\Lambda = diag(e^{j\omega_h} I_{n_h}, e^{j\omega_h} I_{n_v})$ | Frequency domain for discrete 2D Roesser Model |
| (e) | $\begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} : Q > 0, P \in \mathbb{H}_n$ | $\Lambda = (j/\omega)I : \|\omega\| \geq \omega_0$ | High frequency range for continuous time systems |

Table 2.1: Examples of frequency regions $\mathcal{R}(\boldsymbol{\Theta})$.

| | $\boldsymbol{\Pi}$ | Uncertainty region $\mathcal{R}(\boldsymbol{\Pi})$ | Applications |
|---|---|---|---|
| (f) | $\begin{bmatrix} \tau I & 0 \\ 0 & -\tau I \end{bmatrix} : \tau > 0$ | $\Delta \in \mathbb{C}^{n \times n} : \\|\Delta\\| \leq 1$ | Small gain uncertainties |
| (g) | $\begin{bmatrix} \Pi_d & 0 \\ 0 & -\Pi_d \end{bmatrix} :$ $\Pi_d = diag(\tau_1 I, \cdots, \tau_r I)$ $\tau_i > 0 \ \forall i = 1, \cdots, r$ | $\Delta = diag(\Delta_1, \cdots, \Delta_r) :$ $\\|\Delta_i\\| \leq 1 \ \forall i = 1, \cdots, r$ | Structured uncertainties |
| (h) | $\begin{bmatrix} 0 & \tau I \\ \tau I & 0 \end{bmatrix} : \tau > 0$ | $\Delta \in \mathbb{C}^{n \times n}, \Delta + \Delta^* \geq 0$ | Positive real uncertainties |

Table 2.2: Examples of uncertainty regions $\mathcal{R}(\boldsymbol{\Pi})$.

Consider the determinant of the upper left $2 \times 2$ matrix. Notice that

$$\det \begin{bmatrix} 2(1+\epsilon)Re(s_1) & s_2 + \bar{s}_1 \\ s_1 + \bar{s}_2 & 2(1+\epsilon)Re(s_2) \end{bmatrix} = 4(2\epsilon + \epsilon^2)Re(s_1)Re(s_2) - |s_1 - s_2|^2 < 0$$

for sufficiently small $\epsilon > 0$. This implies that there exists $P > 0$ such that (2.3) is not positive semidefinite. This means that $S$ has to be of the form of $S = sI : s \in \bar{\mathbb{C}}_+$. ∎

Few more examples are summarized in Table 2.1 and 2.2. Notice that Table 2.1 shows regions for the frequency variables that are used for the analyses of various type of systems, while Table 2.2 shows regions that are typically used to represent various types of system uncertainties. In the former cases, variables $P$ and $Q$ are conventionally interpreted as the Lyapunov functions, while in the latter cases $\boldsymbol{\Pi}$ can be viewed as a parameterizing set of IQCs. Defining the frequency variables in $\mathbb{C}^{n \times n}$ enables us to express these different objects in a single framework.

## 2.4 Operator-oriented approach vs. Signal-oriented approach

Our approach to robust stability and performance analysis is to rewrite the original problem into a question about well-posedness, in which we study an interaction of a matrix $M$ and a set of uncertain matrices $\Omega$. Alternative approach is purely based on input-output behavior of uncertain operators in $\Omega$, without explicitly defining $\Omega$. For a pair of input and output vectors $(w, x) \in \mathcal{C}^{2n}$, let $(w, x) \in IQC(\mathbf{\Theta})$ mean that

$$\begin{bmatrix} w \\ x \end{bmatrix}^* \Theta \begin{bmatrix} w \\ x \end{bmatrix} \geq 0 \; \forall \Theta \in \mathbf{\Theta}.$$

For example, suppose we want to analyze the stability of $\dot{x} = Ax$. In the operator-oriented approach, the well-posedness of the interconnection of $A$ and $S \in \mathcal{R}(\mathbf{\Theta})$ is analyzed, where $\mathbf{\Theta}$ is given as in Table 2.1 (a). On the other hand, the IQC approach [26] consider the existence of a nontrivial $(w, x) \in \mathcal{C}^{2n}$ such that $w = Ax$ and $(w, x) \in IQC(\mathbf{\Theta})$. Notice that in the later approach, an additional variable $S$ does not appear. Of course, both approach results in an LMI problem $P > 0, A^T P + PA < 0$ for this simple example. However, the IQC approach is potentially more flexible, since depending on the choice of $\mathbf{\Theta}$, $(w, x) \in IQC(\mathbf{\Theta})$ does not imply the existence of $S \in \mathcal{R}(\mathbf{\Theta})$ such that $x = Sw$. (Consider an example $\mathbf{\Theta} = \{-k \; diag(w^{\perp} w^{\perp *}, x^{\perp} x^{\perp *}) : k > 0\}$). When this happens, it is not possible to write an input-output relation defined by an IQC cannot be expressed as an well-posedness model. Thus, we restrict ourselves to the class of Hermitian sets $\mathbf{\Theta}$ for which our approach based on the well-posedness is equally strong as the IQC approach.

**Definition 2** *A Hermitian set $\mathbf{\Theta}$ is said to be* realizable *if $(w, x) \in IQC(\mathbf{\Theta})$ implies the existence of $S \in \mathcal{R}(\mathbf{\Theta})$ such that $x = Sw$.*

Fortunately both approaches are almost equally powerful because essentially all useful Hermitian sets, including the ones in Table 2.1 and 2.2, are realizable.

## 2.5 Symmetric S-procedure and mutual losslessness property

It is known that the KYP lemma is derived as an application of a mathematical technique known as the S-procedure. In this section, we propose a symmetric formulation of the S-procedure as a prelude step towards the KYP lemma. As a starting point of this reformulation, we already employ a slightly generalized form of the S-procedure proposed in [27] rather than its well-known form. A brief history and the motivation for the generalization of the S-procedure will be discussed in Section 2.9.

Let $\mathbf{\Psi}$ be a convex cone in $\mathbb{H}_n$ and $\Phi \in \mathbb{H}_n$. The S-procedure concerns the relationship between the following conditions.

(I) $\exists \Psi \in \mathbf{\Psi}$ such that

$$\Psi + \Phi < 0. \tag{2.4a}$$

(II) For every nonzero complex vector $\zeta$,

$$(\zeta^* \Psi \zeta \geq 0 \ \forall \Psi \in \mathbf{\Psi}) \Rightarrow \zeta^* \Phi \zeta < 0. \tag{2.4b}$$

It is easy to see the implication (I)⇒(II) always holds. If the other direction holds as well, the S-procedure is said to be *lossless*. Conditions for the S-procedure to be lossless have been studied in many literatures. A popular condition, which has been a basis for modern proofs of the KYP lemma, is called *rank-one separability* [28][21].

**Definition 3** $\mathbf{\Psi} \subset \mathbb{H}_n$ *is said to be* rank-one separable *if* $S(\mathbf{\Psi})$ *is equal to the convex hull of* $S_1(\mathbf{\Psi})$ *where*

$$S(\mathbf{\Psi}) := \{X \in \mathbb{H}_n : X \geq 0, X \neq 0, tr \Psi X \geq 0 \ \forall \Psi \in \mathbf{\Psi}\}$$
$$S_1(\mathbf{\Psi}) := \{\zeta\zeta^* \in \mathbb{H}_n : \zeta \in \mathbb{C}^n, \zeta \neq 0, \zeta^* \Psi \zeta \geq 0 \ \forall \Psi \in \mathbf{\Psi}\}.$$

In [21], it is shown that an S-procedure (2.4) with an arbitrary Hermitian matrix $\Phi$ is lossless if and only if $\mathbf{\Psi}$ is rank-one separable. However, it is important to notice that the exact condition can be relaxed if $\Phi$ is known to belong to a restricted class of matrices. For example, the authors of [29] have introduced the notion of *one-vector-lossless sets*, which is weaker than rank-one separability, but sufficient to prove that the S-procedure is lossless for matrices $\Phi \leq 0$.

This indicates that the losslessness property of the S-procedure should be discussed as a matter of the relationship between the set $\mathbf{\Psi}$ and the set $\mathbf{\Phi}$ to which $\Phi$ belongs. To clarify this point, it is natural to introduce the following generalization of the S-procedure, which has a symmetric structure with respect to $\mathbf{\Psi}$ and $\mathbf{\Phi}$:

(P1) $\exists (\Psi, \Phi) \in (\mathbf{\Psi}, \mathbf{\Phi})$ such that $\Psi + \Phi < 0$.

(P2) There does not exist a nonzero complex vector $\zeta$ such that

$$\zeta^* \Psi \zeta \geq 0 \ \forall \Psi \in \mathbf{\Psi} \text{ and } \zeta^* \Phi \zeta \geq 0 \ \forall \Phi \in \mathbf{\Phi}$$

In the original form of the S-procedures (2.4) can be interpreted as a special case of the new formulation in which one of two Hermitian sets is a singleton (up to its positive multiples). In this thesis, we particularly refer to the convex relaxation (P1) above of the condition (P2) the *(symmetric) S-procedure*. The symmetric structure suggests that the condition characterizing losslessness of the new S-procedure above should be a symmetric relationship between $\mathbf{\Psi}$ and $\mathbf{\Phi}$. This relationship can be further studied via dual conditions (negations) of (P1) and (P2).

(D1) There exists a nonzero $X \geq 0$ such that

$$tr\Psi X \geq 0 \; \forall \Psi \in \mathbf{\Psi} \text{ and } tr\Phi X \geq 0 \; \forall \Phi \in \mathbf{\Phi}$$

(D2) There exists a nonzero vector $\zeta$ such that

$$\zeta^* \Psi \zeta \geq 0 \; \forall \Psi \in \mathbf{\Psi} \text{ and } \zeta^* \Phi \zeta \geq 0 \; \forall \Phi \in \mathbf{\Phi}$$

To see that (D1) is the negation of (P1), one can use the Hahn-Banach separation theorem [30] as follows. Consider the following convex cones in $\mathbb{H}_n$.

$$\mathcal{N} = \{N \in \mathbb{H}_n : N < 0\}; \;\; \mathcal{M} = \{\Psi + \Phi : \Psi \in \mathbf{\Psi}, \Phi \in \mathbf{\Phi}\}.$$

Since $\mathcal{N}$ is open in the standard topology on $\mathbb{H}_n$, and the negation of (P1) means that $\mathcal{N}$ and $\mathcal{M}$ are disjoint, there exists a nonzero Hermitian matrix $X$ and $r \in \mathbb{R}$ such that

$$trNX < r \leq trMX \; \forall (N, M) \in (\mathcal{N}, \mathcal{M}).$$

Since $\mathcal{N}$ and $\mathcal{M}$ are cones, we can take $r = 0$ without loss of generality. Then the above inequalities mean that there exists a nonzero $X \geq 0$ such that

$$tr(\Psi + \Phi)X \geq 0 \; \forall (\Psi, \Phi) \in (\mathbf{\Psi}, \mathbf{\Phi}).$$

This implies condition (D1). (Pick $\Phi = 0$ to obtain the first condition of (D1), and $\Psi = 0$ for the second condition of (D1).) The other direction (D1)$\Rightarrow \neg$(P1) is easy to see. Thus (D1)$\Leftrightarrow \neg$(P1).

**Definition 4** $\mathbf{\Psi}$ *and* $\mathbf{\Phi}$ *are said to be* mutually lossless *if (P1)$\Leftrightarrow$(P2), or equivalently (D1)$\Leftrightarrow$(D2). Namely, the symmetric S-procedure is lossless if and only if* $\mathbf{\Psi}$ *and* $\mathbf{\Phi}$ *are* mutually lossless.

## 2.6 Mutual Losslessness Property and the KYP Lemma

The KYP lemma can be viewed as a direct application of the S-procedure. In the study of dynamical systems whose transfer function is given by $\hat{G} = C(sI - A)^{-1}B + D$, $\mathbf{\Psi}$ and $\mathbf{\Phi}$ above are often chosen to be

$$\mathbf{\Psi} = \left\{ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \Theta \in \mathbf{\Theta} \right\}, \quad \mathbf{\Phi} = \left\{ \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \Pi \in \mathbf{\Pi} \right\} \tag{2.5}$$

where $\mathbf{\Theta} \in \mathbb{H}_{2n}, \mathbf{\Pi} \in \mathbb{H}_{2m}$ are convex cones. Now we are in position of formulating the KYP lemma as a systematic way to verify well-posedness via a convex program. Apparently, the following form of the KYP lemma emphasizes the symmetric structure observed in Section 2.1.

**Theorem 1** *(KYP Lemma) Let $\mathbf{\Theta}$ and $\mathbf{\Pi}$ be realizable Hermitian sets. Then (I)⇒(II) holds. Moreover, (I)⇔(II) holds if and only if $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are mutually lossless.*

*(I) There exists $\Theta \in \mathbf{\Theta}$ and $\Pi \in \mathbf{\Pi}$ such that*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0. \tag{2.6}$$

*(II) The interconnection $[M, \mathbf{\Omega}]$ is well-posed, where*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} \mathcal{R}(\mathbf{\Theta}) & 0 \\ 0 & \mathcal{R}(\mathbf{\Pi}) \end{bmatrix}.$$

Notice that condition (I) is verified by a conic program. Typically, this condition can be written in the form of LMIs, whose feasibility can be checked by polynomial time algorithms such as semidefinite programming (SDP). On the other hand, condition (II) means that

$$\left\{ (S_1, S_2) : S_1 \in \mathcal{R}(\mathbf{\Theta}), S_2 \in \mathcal{R}(\mathbf{\Pi}), \det\left( I - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \right) = 0 \right\}$$

is empty. There does not necessarily exist a polynomial time algorithm to check it, although checking the emptiness of a semialgebraic set is known to be decidable [2]. Theorem 1 claims that (I) gives a sufficient condition to guarantee the emptiness of the above set for any combination of $\mathbf{\Psi}$ and $\mathbf{\Phi}$. Moreover, the efficiently verifiable condition (I) is equivalent to the emptiness of the above set if and only if $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are mutually lossless.

**Proof:** By the Hahn-Banach separation theorem, the negation of (I) is the following condition:

(I') There exists a nonzero matrix $X \geq 0$ such that

$$tr \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} X \geq 0 \; \forall \Theta \in \mathbf{\Theta}, \quad tr \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} X \geq 0 \; \forall \Pi \in \mathbf{\Pi}.$$

This is a necessary condition for the following (II'), and (I')⇔(II') holds if and only if $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are mutually lossless.

(II') There exists a nonzero vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^* \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \; \forall \Theta \in \mathbf{\Theta}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^* \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \; \forall \Pi \in \mathbf{\Pi}.$$

Hence it is left to prove that

$$\text{(II')} \;\Leftrightarrow [M, \mathbf{\Omega}] \text{ is ill-posed.}$$

(⇒): By writing $w_1 = Ax_1 + Bx_2$, $w_2 = Cx_1 + Dx_2$, (II') means

$$\begin{bmatrix} w_1 \\ x_1 \end{bmatrix}^* \Theta \begin{bmatrix} w_1 \\ x_1 \end{bmatrix} \geq 0 \; \forall \Theta \in \mathbf{\Theta}, \quad \begin{bmatrix} w_2 \\ x_2 \end{bmatrix}^* \Pi \begin{bmatrix} w_2 \\ x_2 \end{bmatrix} \geq 0 \; \forall \Pi \in \mathbf{\Pi}.$$

Namely, $(w_1, x_1) \in IQC(\mathbf{\Theta})$ and $(w_2, x_2) \in IQC(\mathbf{\Pi})$. By the realizability assumption on $\mathbf{\Theta}$ and $\mathbf{\Pi}$, there exists $S_1 \in \mathcal{R}(\mathbf{\Theta})$ and $S_2 \in \mathcal{R}(\mathbf{\Pi})$ such that $x_1 = S_1 w_1$ and $x_2 = S_2 w_2$. This leads to

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$I$}} & -A & -B \\ & & -C & -D \\ \hline -S_1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$I$}} \\ 0 & -S_2 & & \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \\ x_1 \\ x_2 \end{bmatrix} = 0, \quad \begin{bmatrix} w_1 \\ w_2 \\ x_1 \\ x_2 \end{bmatrix} \neq 0.$$

This implies that $[M, \mathbf{\Omega}]$ is not well-posed.

(⇐): The ill-posedness of $[M, \mathbf{\Omega}]$ implies that

$$\inf_{\Omega \in \mathbf{\Omega}} \left\| \begin{bmatrix} I & -M \\ -\Omega & I \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} \right\| = 0 \text{ for some } \begin{bmatrix} w \\ x \end{bmatrix} \neq 0.$$

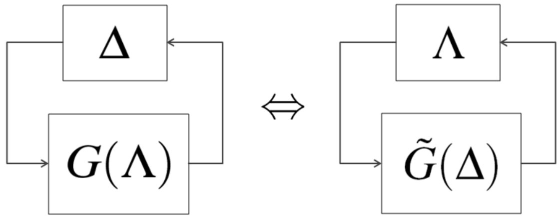


Figure 2.1: Two different ways to interpret the LMI condition (2.6)

Such vectors $(w, x)$ must satisfy $w = Mx$. Hence, the above condition implies $\inf_{\Omega \in \mathbf{\Omega}} \|(I - \Omega M)x\| = 0$ for some $x \neq 0$. Therefore, there exist sequences $\{S_1^{(k)}\} \in \mathcal{R}(\mathbf{\Theta})$, $\{S_2^{(k)}\} \in \mathcal{R}(\mathbf{\Pi})$ such that

$$\left(I - \begin{bmatrix} S_1^{(k)} & 0 \\ 0 & S_2^{(k)} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1^{(k)} \\ u_2^{(k)} \end{bmatrix}$$

and $\lim_{k\to\infty} u_1^{(k)} = 0$, $\lim_{k\to\infty} u_2^{(k)} = 0$. Since $S_1^{(k)} \in \mathcal{R}(\mathbf{\Theta})$ for each $k$,

$$\lim_{k\to\infty} w_1^* \begin{bmatrix} I \\ S_1^{(k)} \end{bmatrix}^* \Theta \begin{bmatrix} I \\ S_1^{(k)} \end{bmatrix} w_1 \geq 0, \ \forall \Theta \in \mathbf{\Theta}, \forall w_1.$$

In particular, by taking $w_1 = Ax_1 + Bx_2$, and noticing that $S_1^{(k)}(Ax_1 + Bx_2) = x_1 - u_1^{(k)}$, we have

$$\lim_{k\to\infty} w_1^* \begin{bmatrix} I \\ S_1^{(k)} \end{bmatrix}^* \Theta \begin{bmatrix} I \\ S_1^{(k)} \end{bmatrix} w_1 = \lim_{k\to\infty} \begin{bmatrix} Ax_1 + Bx_2 \\ x_1 - u_1^{(k)} \end{bmatrix}^* \Theta \begin{bmatrix} Ax_1 + Bx_2 \\ x_1 - u_1^{(k)} \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^* \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \ \forall \Theta \in \mathbf{\Theta}.$$

Therefore we obtained the first condition of (II'). Similarly, noticing that $S_2^{(k)} \in \mathcal{R}(\mathbf{\Pi})$ for each $k$ and using the equality $S_2^{(k)}(Cx_1 + Dx_2) = x_2 - u_2^{(k)}$, we obtain the second condition of (II'). ■

In many literatures (e.g., [28][21]), the KYP lemma is stated between a matrix inequality condition such as (2.6) and a frequency domain inequality (FDI). Theorem 1, on the other hand, deals with the well-posedness condition directly instead of an FDI, since in many robust stability and performance analyses, the desired system theoretic property can be written as a well-posedness condition.

The KYP lemma, stated in the form of Theorem 1, allows two different interpretations (Figure 2.1). Assuming that a transfer function $G(\Lambda) = C(I - \Lambda A)^{-1}\Lambda B + D$ is well-defined on $\mathcal{R}(\mathbf{\Theta})$. In this interpretation, $\Lambda = s^{-1}I$ is the frequency variable. Then the LMI (2.6) means that the interconnection of $G(\Lambda)$ and an uncertainty $\Delta$ is well-posed. It is also possible to consider a transfer function $\tilde{G}(\Delta) = B(I - \Delta D)^{-1}\Delta C + A$. In this interpretation, the LMI (2.6) means that the interconnection of $\tilde{G}(\Delta)$ and $\Lambda$ is well-posed. This implies that $\mathcal{R}(\mathbf{\Theta})$ and $\mathcal{R}(\mathbf{\Pi})$ play symmetric roles, although conventionally the former is considered as the frequency domain and the latter is considered as the space of uncertainties.

## 2.7 Symmetric view on the KYP Lemma

Theorem 1 has a clear symmetric structure with respect to $\Theta$ and $\Pi$. In the next examples, we show that two seemingly different questions are in fact the same question, after noticing this symmetry.

### 2.7.1 Continuous vs. discrete time models and positive realness vs. bounded realness

The bounded real lemma and the positive real lemma are the two most important special cases of the KYP lemma. LMI conditions for positive realness and the bounded realness can be obtained by choosing $\Pi$ in (2.6) as

$$\text{Bounded realness: } \Pi_b = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \text{Positive realness: } \Pi_p = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

In each case, continuous and discrete time analysis correspond to the choice of $\Theta$ as

$$\text{Discrete time: } \Theta_d = \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}, \quad \text{Continuous time: } \Theta_c = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}.$$

With the symmetry considered in Section 2.7 in mind, consider the congruence transformation $T^*\Theta_c T = \Theta_d$ where

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}.$$

This operations amounts to the Möbius transformation $s = \frac{z-1}{z+1}$ between frequency variables, and converts the continuous-time frequency domain into the discrete-time one ((a) and (c) in Table 2.1). The same transformation $T^*\Pi_p T = \Pi_b$ applied to the other term can be viewed as a conversion of the small gain IQC into the passivity IQC ((f) and (h) in Table 2.2). The meaning of these two operations are flipped when the LMI (2.6) is interpreted through system $\tilde{G}$ instead of $G$ (Figure 2.1).

### 2.7.2 Diagonal Lyapunov functions vs. structured singular values

Distributed control has been actively studied in the past few decades. It is known, however, that a certain class of control design with imperfect information is NP-hard [2]. Hence, most of the research effort has been made to uncover the class of distributed control problems for which a tractable algorithm is available to obtain the solution, or at least to reasonably approximate the solution with minimal conservatism [31, 32, 33, 34, 32]. A popular approach, especially in the LMI-based synthesis for distributed state feedback control, is to use *diagonal* quadratic control-Lyapunov functions. (If a stable linear dynamical system has a diagonal Lyapunov function, the system is said to be *diagonally stable*.) If the control-Lyapunov function is assumed to be diagonal, then a simple but critically important technique of change of variables is available that converts the original Bilinear Matrix Inequality (BMI) condition into an LMI condition (we will review this technique in Section 4.5.2). In this way, a distributed state feedback synthesis is obtained by a tractable algorithm.

Of course, requiring diagonal stability rather than mere stability is solely for the sake of computational benefit in the control design process, and there is no physical justification to do so. As a result, this assumption becomes the source of conservatism in the distributed control design. Therefore, a quantitative understanding of the conservatism gap introduced by the use of diagonal Lyapunov functions is of natural interest.

The symmetric interpretation of parametrized Lyapunov functions and IQCs allows us to relate this question to the structured singular value problems. More precisely, the following two seemingly different problems are in fact precisely the same problem.

Q1: Suppose an $m$-input-$m$-output discrete time system $G$ given by

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k)$$

is interconnected to the scalar uncertainty $u(k) = \delta y(k)$, $|\delta| < 1$. The internal stability of this interconnection is guaranteed if there exist positive definite (full) matrices $P$ and $Q$ satisfying (2.1). One may wish to have not only stability but also diagonal stability by additionally restricting $P$ to be diagonal, possibly for the distributed control purpose as discussed above. How much more restrictive is it to require diagonal stability rather than mere stability?

Q2: Suppose an $m$-input-$m$-output discrete time system $H$ given by

$$x(k+1) = Dx(k) + Cu(k), \quad y(k) = Bx(k) + Au(k)$$

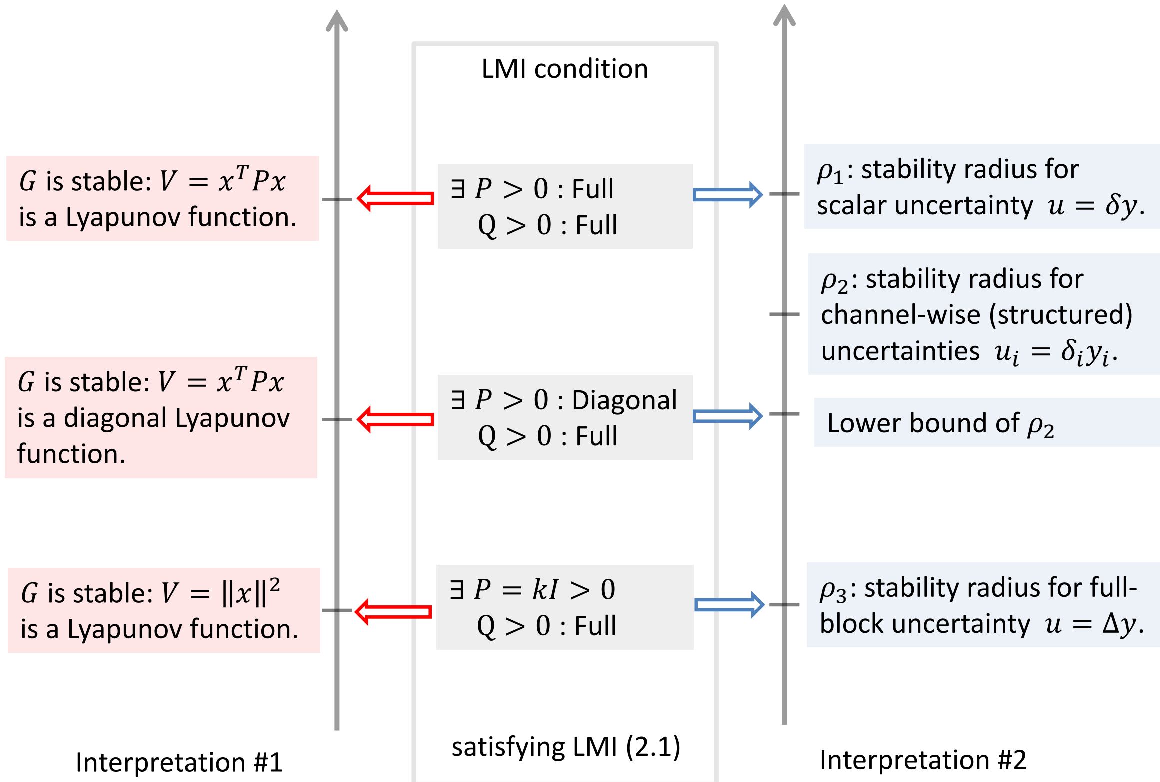


Figure 2.2: Connections between diagonal stability conditions and the shape of uncertainty.

is interconnected to the scalar uncertainty $u(k) = \delta y(k)$, $|\delta| < \rho$. The existence of positive definite (full) matrices $P$ and $Q$ satisfying (2.1) guarantees that the stability radius is upper bounded by 1. Now, suppose that the uncertainty constant is allowed to be different on each channel, i.e., $u_i(k) = \delta_i y_i(k)$, $|\delta_i| < \rho$. To see if 1 remains to be an upper bound of the stability radius, one may apply the diagonal scaling technique to compute the upper bound of the structured singular values. This means that one additionally restrict $P$ to be diagonal and check the feasibility of the LMI (2.1). How much more difficult to prove that 1 is an upper bound of the stability radius for the scaler uncertainty $u(k) = \delta y(k)$ as compared to the channel-wise (diagonally structured) uncertainty $u_i(k) = \delta_i y_i(k)$?

This observation may suggest that the study of conservatism in robust control theory can be directly applied to the conservatism study for the distributed control theory.

## 2.8 Generalization of the mutual losslessness

So far, we have defined the notion of mutual losslessness as a relationship between two sets of Hermitian matrices. However, there is a straightforward generalization of the definition to the relationship among multiple sets of Hermitian matrices as follows. Suppose $N$-tuple of sets of Hermitian matrices $\boldsymbol{\Psi_1}, \cdots, \boldsymbol{\Psi_N} \subset \mathbb{H}_n$ are all convex cones. Then the $N$-tuple of convex cones $(\boldsymbol{\Psi_1}, \cdots, \boldsymbol{\Psi_N})$ is said to be mutually lossless if the following two conditions are equivalent:

(D1') There exists a nonzero $X \geq 0$ such that

$$tr\Psi_1 X \geq 0 \; \forall \Psi_1 \in \boldsymbol{\Psi_1}, \;\; tr\Psi_2 X \geq 0 \; \forall \Psi_2 \in \boldsymbol{\Psi_2}, \;\; \cdots \;\; tr\Psi_N X \geq 0 \; \forall \Psi_N \in \boldsymbol{\Psi_N}.$$

(D2') There exists a nonzero vector $\zeta$ such that

$$\zeta^* \Psi_1 \zeta \geq 0 \; \forall \Psi_1 \in \boldsymbol{\Psi_1}, \;\; \zeta^* \Psi_2 \zeta \geq 0 \; \forall \Psi_2 \in \boldsymbol{\Psi_2}, \;\; \cdots \;\; \zeta^* \Psi_N \zeta \geq 0 \; \forall \Psi_N \in \boldsymbol{\Psi_N}.$$

If each of $\boldsymbol{\Psi_i}$ are finitely generated, i.e., it is a convex hull of a finite number of rays $\{kM_{ij} : k \geq 0, M_{ij} \in \mathbb{H}_n\}$, then without less of generality, the above condition can be written in such a way that $N$ is finite and each of $\boldsymbol{\Psi_i}$ is a ray of a Hermitian matrix.

## 2.9 Discussion

### 2.9.1 Some history of the S-procedure

A theoretical key component of the S-procedure can be traced back to the work of O. Toeplitz and F. Hausdorff on the numerical range of a square matrix in the years 1918-1919. Given two Hermitian matrices $\Psi_1, \Psi_2 \in \mathbb{H}_n$ and a vector function

$$\pi(z) = [\sigma_1(z) \;\; \sigma_2(z)], \;\; \sigma_i(z) = z^* \Psi_i z, \tag{2.7}$$

the Hausdorff-Toeplitz theorem implies that the set $\{\pi(z) : \|z\| = 1, z \in \mathbb{C}^n\}$ is convex in $\mathbb{R}^2$. Indeed, a stronger result is known today for the complex field: such a set is convex in $\mathbb{R}^3$ for three Hermitian forms and $n > 2$. For the real field, Dines proved in 1941 that the set $\{\pi(z) : z \in \mathbb{R}^n\}$ is a convex cone, when $\Psi_1$ and $\Psi_2$ in (2.7) are real symmetric matrices. In 1971, Yakubovich [35] used these results to prove that the following form of the S-procedure is lossless when $\mathcal{H} = \mathbb{R}^n$ and $N = 2$, or $\mathcal{H} = \mathbb{C}^n$ and $N = 3$:

(I) There exists $\tau_k \geq 0,\ k = 2, \cdots, N$ such that

$$\sigma_1(z) + \sum_{k=2}^{N} \tau_k \sigma_k(z) \leq 0, \quad \forall z \in \mathcal{H} \tag{2.8a}$$

(II) There is no $z \in \mathcal{H}$ such that

$$\sigma_1(z) > 0 \text{ and } \sigma_k(z) \geq 0\ \forall k = 2, \cdots, N \tag{2.8b}$$

.

The above convexity results play a crucial role in his proof, so that the geometric separation of the image of $\pi(z)$ and the positive orthant implies the existence of a separating hyperplane $z_1 + \tau_2 z_2 + \cdots + \tau_N z_N = 0$. Much later, the S-procedure on the real field is proved to be lossless for $N = 3$ by Polyak [36] by making an additional assumption, using the convexity result obtained by Brickman.

The S-procedure became popular in the system and control theory since Yakubovich's pioneering work in this research domain. However, it is known that a similar method is used in the control theory by Lur'e and Postnikov [37] as early as 1944. In 1990's, Megretski and Treil [14] proved that the S-procedure is lossless for any finite number $N$ on the infinite dimensional space $\mathcal{H} = L_2$. More thorough coverage of the history of the S-procedure and its applications can be found in [22, 38, 39] and references therein.

### 2.9.2 What's new about the mutual losslessness?

In a slightly different form, the S-procedure can be expressed as:

(I) $\exists \Gamma \in \mathbf{\Gamma}$ such that $\Gamma < 0$.

(II) There does not exist $\zeta \in \mathcal{H}, \zeta \neq 0$ such that $\zeta^* \Gamma \zeta \geq 0\ \forall \Gamma \in \mathbf{\Gamma}$

where $\mathbf{\Gamma}$ is a convex cone in the space of Hermitian matrices. Notice that the S-procedure of the form of (2.8) corresponds to the case in which $\mathbf{\Gamma}$ is a cone $\mathbf{\Gamma} = \{\sum_{k=1}^{N} \tau_k \Psi_k : \tau_k \geq 0\}$ generated by a set of given finite set of Hermitian matrices $\Psi_k$. Namely, $\mathbf{\Gamma}$ is a polyhedral cone in the space of matrices.

However, we have already seen in previous sections that the set $\mathbf{\Gamma}$ appearing in the simplest system analysis is already not polyhedral. For example, in the dissipativity analysis (2.1), $\mathbf{\Gamma}$ can be written as a Minkowski sum $\mathbf{\Gamma} = \mathbf{\Psi} + \mathbf{\Phi}$ of two convex cones

$$\boldsymbol{\Psi} = \left\{ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : P > 0 \right\}, \quad \boldsymbol{\Phi} = \left\{ \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : Q > 0 \right\}.$$

Namely, each of them is a *spectrahedral* cone. Similarly, many of the convex cones we frequently use in the well-posedness analysis are in fact *spectrahedra* rather than polyhedra, because they are often parametrized by positive definite matrices.

It is known by the Minkowski-Weyl Theorem [40] that a convex cone in $\mathbb{R}^n$ is finitely generated if and only if it is a polyhedron. This means that, if one wants to formulate the well-posedness analysis in the classical framework of the S-procedure (2.8), the number of constraints $N$ often has to be infinity. This gave a motivation in [27] to consider more general cones as the constraint set in the S-procedure. This leads to the formulation (2.4) from which we have started our discussion. Notice that in (2.4), the set $\boldsymbol{\Psi}$ need not be polyhedral. The notion of mutual losslessness is obtained as the further generalization in this direction to emphasize the symmetric relationships among multiple convex cones. In principle, the notion of mutual losslessness can be used in a broader context than the KYP lemma. For example, it can be used to describe the conservatism in the uncertain semidefinite programs considered in [41].

## 2.10 Special Cases

### 2.10.1 Finite frequency KYP lemma

Let $\hat{G}(s) = C(sI - A)^{-1}B + D$ be a stable transfer function, and $\gamma$ be the largest gain of the frequency response over a finite frequency range $0 \leq \omega_1 \leq \omega \leq \omega_2 \leq \infty$, i.e.,

$$\gamma = \sup_{\omega \in [\omega_1, \omega_2]} \|\hat{G}(j\omega)\|.$$

Notice that an upper bound of $\gamma$ can be confirmed by checking the well-posedness of $[M, \boldsymbol{\Omega}]$, where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & \boldsymbol{\Delta} \end{bmatrix}$$

$$\boldsymbol{\Lambda} = \{\frac{j}{\omega} I : \omega_1 \leq |\omega| \leq \omega_2\}, \quad \boldsymbol{\Delta} = \{\Delta \in \mathbb{C}^{m \times m} : \|\Delta\| \leq 1/\gamma\}.$$

The above domain $\boldsymbol{\Delta}$ can be expressed as $\boldsymbol{\Delta} = \mathcal{R}(\boldsymbol{\Pi})$ where $\boldsymbol{\Pi}$ is chosen as in (f) in Table 2.2. One can also find an appropriate $\boldsymbol{\Theta}$ so that $\boldsymbol{\Lambda} = \mathcal{R}(\boldsymbol{\Theta})$. A choice of such $\boldsymbol{\Theta}$ for the case with $\omega_2 = \infty$ (one is interested in the entire frequency region higher than $\omega_1$) is given in (e) in Table 2.1, but it is also possible to find a corresponding $\mathbf{\Theta}$ for the low frequency range ($0 \leq \omega \leq \omega_2 < \infty$) and middle frequency range ($0 < \omega_1 \leq \omega \leq \omega_2 < \infty$). By applying the KYP lemma (Theorem 1), one can derive a convex program whose feasibility guarantees well-posedness. However, it is not immediately clear whether this KYP lemma is lossless.

A breakthrough result is given by the theory of the Generalized KYP lemma [21, 27]. In [27], it is shown that the set of Hermitian matrices

$$\mathbf{\Psi} = \left\{ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \Theta \in \mathbf{\Theta} \right\}, \quad \mathbf{\Theta} = \left\{ \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} : \begin{array}{c} P \in \mathbb{H}_n \\ Q > 0 \end{array} \right\} \tag{2.9}$$

is in fact rank-one separable[1] (Definition 3). Since rank-one separable set is mutually lossless with any rays of a Hermitian matrix, we have the mutual losslessness between $\mathbf{\Psi}$ and $\mathbf{\Phi}$, where

$$\mathbf{\Phi} = \left\{ \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} : \Pi \in \mathbf{\Pi} \right\}, \quad \mathbf{\Pi} = \{\kappa \Pi_0 : \kappa > 0\}\,. \tag{2.10}$$

Since (2.10) includes (e) in Table 2.1 as a special case, the above gain analysis over a finite frequency range can be performed without conservatism.

Unfortunately, our framework does not provide an alternative proof for this specific mutual losslessness so we have to "borrow" a proof from the original papers [21, 27]. Our purpose of revisiting this example was to show that the exactness of the finite frequency KYP lemma can be understood via the general frameworkof mutual losslessness.

### 2.10.2 Image processing

When the mutual losslessness does not hold between $\mathbf{\Psi}$ and $\mathbf{\Phi}$, Theorem 1 holds only in one direction, i.e., the LMI test is only a sufficient condition for the well-posedness. Let us consider such a situation in an image processing example. Due to the two dimensional nature of image signals, it is intuitive to consider every process of image manipulation as a mapping from the input image $u(i, j)$ to the output image $y(i, j)$, which are both 2-dimensional signals with vertical and horizontal coordinate $(i, j)$. In his paper [42] in 1975, Robert Roesser proposed an attractive linear state space

[1] Although (2.9) corresponds to the analysis of high frequency region, it is also shown that the rank-one separability holds for the low frequency analysis and the middle frequency analysis as well.

model for linear image processing, which is now known as 2-D model, n-D model, or the Roesser's model.

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j)$$
$$y(i,j) = C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j)$$

Two portions of the state variable, $x^h(i,j)$ and $x^v(i,j)$, are to deliver information horizontally and vertically. In one view, this is a natural extension of the state space model from the first order linear ordinary differential equation to the first order linear partial differential equation. Image restoration methods from noisy observations are proposed in [43] by applying Kalman filter generalized to two dimensional settings. Other authors [44] concerns the worst case gain of the above system, and proposes $H_\infty$ filtering and $H_\infty$ control methods. Suppose one wants to ensure that the energy of the output image is bounded by the energy of the input image, i.e., $\|y\| \leq \|u\|$. It is a straightforward application of the main loop theorem that this is equivalent to saying that the interconnection $[M, \mathbf{\Omega}]$ is well-posed, where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{\Lambda} & 0 \\ 0 & \mathbf{\Delta} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \left\{ \begin{bmatrix} e^{-j\omega_h} I & 0 \\ 0 & e^{-j\omega_v} I \end{bmatrix} : \omega_h, \omega_v \in \mathbb{R} \right\}, \quad \mathbf{\Delta} = \{\Delta \in \mathbb{C}^{m\times m} : \|\Delta\| \leq 1\}.$$

In the above, $e^{-j\omega_h}$ and $e^{-j\omega_v}$ are the (spatial) frequency variables for the discrete Fourier transform in the horizontal and the vertical coordinates. Notice that Theorem 1 has a flexibility to perform the well-posedness analysis for this system. The KYP lemma for 2D-systems considered in [25] is obtained as a special case of Theorem 1 by properly specifying $\mathbf{\Theta}$ as the item (d) in Table 2.1 in p.16. However, if the set $\mathbf{\Psi}$ in (2.5) is defined using this $\mathbf{\Theta}$, and the set $\mathbf{\Phi}$ is defined using a ray $\mathbf{\Pi} = \{\tau\Pi_0 : \tau > 0\}$ of a fixed Hermitian matrix $\Pi_0$, then the mutual losslessness does not hold between $\mathbf{\Psi}$ and $\mathbf{\Phi}$ in general. This is essentially the reason why the LMI test proposed in [25] is only a sufficient condition.

### 2.10.3 Large-scale dynamical systems

The 2-D Roesser's model appears in a slightly different context as well. Consider a situation where identical dynamical systems are interconnected to each other in the spatially invariant manner as in Figure 2.3. Such a model naturally arises in the study of dynamical behavior of a vehicle platoon, a constellation of satellites, and the formation flight

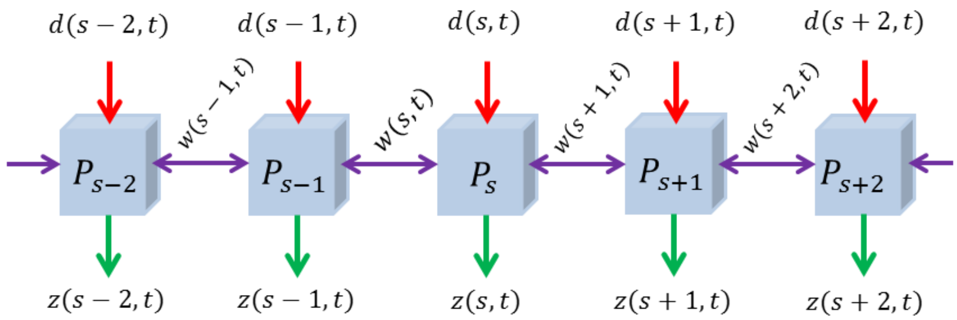


Figure 2.3: Spatially invariant systems

of micro UAVs. It also shows up as a discrete approximation of distributed parameter systems. State space of such systems evolves in time directions as well as the spatial direction. The evolution in the time direction is indexed by $t \in \mathbb{R}$, while the evolution in the spatial direction is indexed by $s \in \mathbb{Z}$. If each unit has an input channel $d(s,t)$ and an output channel $z(s,t)$, the state space is modeled as the following 2-D system:

$$\begin{bmatrix} \dot{x}(t,s) \\ w(t,s+1) \end{bmatrix} = A \begin{bmatrix} x(t,s) \\ w(t,s) \end{bmatrix} + Bd(t,s)$$

$$z(t,s) = C \begin{bmatrix} x(t,s) \\ w(t,s) \end{bmatrix} + Dd(t,s)$$

As in the previous example, the $H_\infty$ performance of this systems is analyzed via the well-posedness of the interconnection $[M, \mathbf{\Omega}]$, where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{\Lambda} & 0 \\ 0 & \mathbf{\Delta} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \left\{ \begin{bmatrix} sI & 0 \\ 0 & e^{-j\omega} I \end{bmatrix} : s \in \bar{\mathbb{C}}_+, \omega \in \mathbb{R} \right\}, \quad \mathbf{\Delta} = \{\Delta \in \mathbb{C}^{m\times m} : \|\Delta\| \leq 1\}.$$

In the above, $s$ is the frequency variable for the Laplace transform of $x(t,s)$ (as a function of time $t$), and $e^{-j\omega}$ is the frequency variable for the discrete Fourier transform of $x(t,s)$ (as a function of space $s$). One can find a set of Hermitian matrices $\mathbf{\Theta}$ such that $\mathbf{\Lambda} = \mathcal{R}(\mathbf{\Theta})$. Unfortunately, as in the previous example the mutual losslessness does not hold between the corresponding $\mathbf{\Psi}$ and $\mathbf{\Phi}$ in general, and the KYP lemma yields an LMI condition which is only sufficient for the well-posedness.

### 2.10.4 $\mu$-analysis

The problem of computing the structured singular value $\mu$ of a given complex matrix is NP-hard [45, 15]. The gap between $\mu$ and its small-gain based upper bound can be understood with a combination of Hermitian sets $\mathbf{\Psi}$ and $\mathbf{\Phi}$ that fail to satisfy the strong mutual losslessness property. The structured singular value of $G \in \mathbb{C}^{m\times m}$ is defined by

$$\mu_{\Pi}(G) = \frac{1}{\min\{|\tau| : det(I - \tau\Delta G) = 0, \Delta \in \mathcal{R}(\mathbf{\Pi})\}}.$$

Now consider the problem of determining whether

$$\sup_{\lambda\in\bar{\mathbb{C}}_+} \mu_{\Pi}(\hat{G}(s)) < 1. \tag{2.11}$$

Condition (2.11) is indeed the well-posedness condition of the interconnection $[M, \mathbf{\Omega}]$ where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} \mathcal{R}(\mathbf{\Theta}) & 0 \\ 0 & \mathcal{R}(\mathbf{\Pi}) \end{bmatrix}$$

in which $\mathbf{\Theta}$ and $\mathbf{\Pi}$ are specified by item (a) in Table 2.1 and item (g) in Table 2.2. Thus it can be readily seen from Theorem 1 that the existence of $\Theta \in \mathbf{\Theta}$ and $\Pi \in \mathbf{\Pi}$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Theta \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0 \tag{2.12}$$

is a sufficient condition for (2.11). However, the mutual losslessness does not hold in general between the sets of Hermitian matrices $\mathbf{\Psi}$ and $\mathbf{\Phi}$ defined as in (2.5) using this choice of $\mathbf{\Theta}$ and $\mathbf{\Pi}$. Hence (2.12) is only a sufficient condition for (2.11), and the LMI test (2.12) only corresponds to computing a convex upper bound of $\mu$ using a diagonal scaling technique. This is understandable given the computational difficulty of $\mu$.

### 2.10.5 Bounded real lemma for positive systems

For positive systems, it is widely known that stability and diagonal stability are equivalent notions. More precisely, a matrix $A$ is Hurwitz if and only if there exists a diagonal matrix $P > 0$ such that $A^T P + PA < 0$ provided $A$ is a Metzler matrix (all off-diagonal entries are non-negative). A proof can be found, for example, in [46]. The diagonal stability can be further extended to the bounded realness test [47]. Suppose that a transfer function $\hat{G}(s) = C(sI - A)^{-1}B + D$ has internally positive realization, i.e., $A$ is Metzler and $B, C, D$ are entry-wise non-negative matrices. Then $\|\hat{G}\|_\infty < 1$ if and only if there exists a diagonal matrix $P > 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0.$$

The losslessness of this bounded real lemma strongly relies on the fact that the corresponding $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are mutually lossless. A complete proof is given in Theorem 8 later.

## 2.11 Conclusion

We proposed a symmetric formulation of the S-procedure and the KYP lemma, and showed that an algebraic condition called *mutual losslessness* among the sets of Hermitian matrices is the exact condition for the S-procedure and the KYP lemma to be lossless. The proposed form of the KYP lemma was shown to have sufficient generality to unify some of the recent extensions of system analysis tools. As a result, the notion of mutual losslessness can explain the lossy and lossless properties of various types of KYP lemmas in a single framework.

However, there is so far no general method for proving the mutual lossless property. Therefore in many particular analysis problems, we need to rely on existing individual techniques to prove losslessness. For example, in order to prove the mutual losslessness for the finite frequency KYP lemma (Section 2.10.1), we still have to rely on the technique in [27]. Finding a practically useful combination of Hermitian sets that satisfies the mutual lossless properties thus remains an important future research direction. Nevertheless, we believe that this approach opens new perspectives for understanding and proving losslessness of the S-procedure and the KYP lemma. It is also an interesting future work to consider how the symmetric formulation of the KYP lemma is related to several different types of recent KYP like lemmas besides the ones considered in 2.10. It will also be necessary to consider further generalizations to accommodate various types of recent results.

## Chapter 3

# Cone-Preserving Transfer Functions

In the previous chapter, we have considered a general method to analyze stability and performance of linear dynamical systems, using the well-posedness model and the KYP lemma. In this chapter, we will focus on a special class of linear dynamical systems for which particularly efficient methods are available to prove their stability and performances. The class of dynamical systems we focus on in this chapter is the class of "cone-preserving dynamical systems". More precisely, we focus on a class of square MIMO transfer functions that map a proper cone in the space of $L_2$ input signals to the same cone in the space of output signals. The main purpose of this chapter is to show that transfer functions in this class have the "DC-dominant" property: the spectral radius of the operator is attained by a DC input signal and hence, the dynamic stability of the interconnected transfer functions is guaranteed solely by the static gain analysis. Using this property, we can also prove delay-independent stability of cone-preserving delay differential equations. An interesting by-product of this consideration is an alternative proof of the delay-independent mean-square stability of a multi-dimensional geometric Brownian motion.

### 3.1 Introduction

In this chapter, we tune our attention to the special class of linear dynamical systems that possess the "cone-preserving" property. For instance, the following examples are of our interest:

(a) (Positive systems): If $A$ is a Metzler matrix (all off-diagonal entries are nonnegative) and $B$ is a square entry-wise nonnegative matrix, the following system defines a map $G : u \mapsto x$ such that $x(t) \geq 0 \ \forall t \geq 0$ as long as $u(t) \geq 0 \ \forall t \geq 0$:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \ x(0) = 0.$$

Thus $G$ leaves the positive orthant in $\mathbb{R}^n$ invariant.

(b) (Covariance dynamics): The following matrix differential equation defines a map $G : U \mapsto X$ such that $X(t) \geq 0 \; \forall t \geq 0$ as long as $U(t) \geq 0 \; \forall t \geq 0$.

$$\frac{d}{dt}X(t) = AX(t) + X(t)A^T + BU(t)B^T, \; X(0) = 0$$

Thus $G$ leaves the positive semidefinite cone in $\mathbb{R}^{n \times n}$ invariant. We will see in Chapter 3 that the above systems represents dynamics of covariance matrix of the multi-dimensional geometric Brownian motion.

It should be noted that both of the above examples belong to what is known as the monotone dynamical systems [49, 50], which are widely studied in various disciplines ranging from pure mathematics to system biology. In this chapter, we particularly focus on their frequency domain property, and uncover their "DC-dominant" property.

Recently, it is pointed out by several papers including [51, 52, 74] that some important features of internally positive transfer functions, the most apparent class of monotone systems, can be captured by looking at their static gains only. For example, if $G$ defines an internally positive systems, the maximum singular value of $\hat{G}(j\omega)$ attains its maximum at $\omega = 0$ and hence its $H_\infty$ norm coincides with the maximum singular value of $\hat{G}(0)$. This peculiar property is certainly valuable, since it eliminates a need for the "frequency sweep" often required for the system analysis in generic theories and allows one to focus only on static gains of the system. In [52], a MIMO transfer function such that $|\hat{G}_{ik}(j\omega)| \leq \hat{G}_{ik}(0)$for all $\omega \in \mathbb{R}$ is referred to as a *positively dominated* system, and its properties are investigated.

This naturally raises a question: Does this property hold for more general cone-preserving linear systems? It turns out that the same $H_\infty$ norm property fails to hold when the positive orthant is replaced with more general cones. As we will see, however, a weaker but similar property remains to hold for more general cone-preserving systems: the spectral radius of $G(j\omega)$ attains its maximum at $\omega = 0$. We refer to this as the DC-dominant property. This can be seen as a generalization of the same property of positive systems to more general cone-preserving systems, in a similar fashion as the Perron-Frobenius theorem regarding positive matrices is extended to the following Krein-Rutman theorem (e.g., [53]):

**Theorem 2** *(Krein-Rutman) Let $G$ be a compact linear operator on a Banach space $X$. Suppose that $GK \subseteq K$, where $K$ is a closed generating cone[1] in $X$. If the spectral radius $\rho(G)$ of $G$ is positive, then there exists a nonzero vector $x \in K$ such that $Gx = \rho(G)x$.*

Not surprisingly, some of our results follow as implications of the Krein-Rutman theorem. The DC-dominant property is compactly stated in Theorem 3 in Section 3.2. This property further implies that the dynamic stability of the

[1] A cone $K$ is said to be generating if $X = K - K$.

feedback interconnection of cone-preserving transfer functions follows solely from the spectral radius condition on their static gains, which is certainly untrue for general linear systems. This observation is summarized in Theorem 4 in Section 3.5 as the "small gain theorem for cone-preserving systems". It serves as the basis for the stability analysis of linear differential equations with cone preserving property. These two theorems are deemed to be the main results in this chapter. It should be noted that these results only depend on input-output properties of cone-preserving operators and, as such, are realization independent.

Of course, the standard theory of linear dynamical system is entirely applicable to the class of stability analysis problems we consider in this chapter. Nevertheless, there is a value in developing a specialized theory for cone preserving linear systems, mainly due to the additional attractive properties that cone preserving systems possess. Moreover, cone-preserving dynamical systems are ubiquitous in real world: dynamics of traffic flow, algorithms involving stochastic matrices such as PageRank, economic study through Leontief's input-output analysis are all naturally modeled as a positive system. Linear systems preserving the positive definite cone routinely appear in statistics, signal processing and filtering algorithms concerned with dynamics of error covariances, as well as in quantum systems [48]. We believe that our main theorem has an advantage over the standard linear systems theory and simplifies the stability analysis of such systems by explicitly exploiting their DC-dominant property.

This chapter is organized as follows: Elements of operator theory and functional analysis are introduced in Section 3.2. Definition and basic results on monotone systems are reviewed in Section 3.3. In Section 3.4, we show the DC-dominant property of cone-preserving transfer functions that will be used to show our main results in Section 3.5. Two examples of cone-preserving systems that preserve the positive orthant and the positive semidefinite cone are shown in Section 3.6. We further investigate the property of "delay-independent" stability of cone-preserving systems in Section 3.7, which gives an alternative proof of the delay-independent mean square stability of the geometric Brownian motion in Section 3.8.

## 3.2 Cone-preserving Operators

Let $\mathbb{R}^n[0,\infty)$ be the vector space of $\mathbb{R}^n$-valued functions defined on $[0,\infty)$. Denote by $\mathcal{L}_2[0,\infty) \subset \mathbb{R}^n[0,\infty)$ the space of square integrable functions

$$\mathcal{L}_2[0,\infty) = \{v \in \mathbb{R}^n[0,\infty) : \int_0^\infty \|v(t)\|^2 dt < +\infty\}.$$

It is well known that there is an isomorphism between the space of linear time invariant operators from $\mathcal{L}_2[0,\infty)$ to itself and the space of transfer functions in the Hardy space $\mathcal{H}_\infty$ (e.g., [13], Theorem 3.30). It can be shown (e.g.,

[12]) that the induced norm is computed by

$$\|G\|_\infty = \sup_{s\in\bar{\mathbb{C}}_+} \|\hat{G}(s)\| = ess\sup_{\omega\in\mathbb{R}} \|\hat{G}(j\omega)\|.$$

In what follows, when the "hat" symbol is used for signals, it represents their Laplace transforms. If "hat" is used for operators, it indicates transfer functions on the Laplace domain. In the above expression, $\hat{G}(s)$ is a complex matrix valued function of $s$.

Let $K \subset \mathbb{R}^n$ be a proper cone[2]. We consider a cone in $\mathcal{L}_2[0,\infty)$ defined by

$$\mathcal{L}_2^K[0,\infty) := \{v \in \mathcal{L}_2[0,\infty) : v(t) \in K \ a.e.\}$$

where "$a.e.$" is respect to the Lebesgue measure on $[0,\infty)$. We also define a space of maps preserving $\mathcal{L}_2^K[0,\infty)$ by

$$\mathcal{H}_\infty^K := \{G \in \mathcal{H}_\infty : G\mathcal{L}_2^K[0,\infty) \subseteq \mathcal{L}_2^K[0,\infty)\}.$$

It is shown in Lemma 3 below that the space $\mathcal{H}_\infty^K$ is complete. Let $P_T$ be the truncation operator defined by

$$(P_t v)(t) = \begin{cases} v(t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}.$$

The extended spaces $\mathcal{L}_{2e}[0,\infty)$ and $\mathcal{L}_{2e}^K[0,\infty)$ are

$$\mathcal{L}_{2e}[0,\infty) = \{v \in \mathbb{R}^n[0,\infty) : P_T v \in \mathcal{L}_2[0,\infty)\ \forall 0 \le T < \infty\}$$
$$\mathcal{L}_{2e}^K[0,\infty) = \left\{v \in \mathbb{R}^n[0,\infty) : P_T v \in \mathcal{L}_2^K[0,\infty)\ \forall 0 \le T < \infty\right\}.$$

Operators in $\mathcal{H}_\infty$ have a natural causal extension to operators from $\mathcal{L}_{2e}[0,\infty)$ to itself ([54], Section 2.4). This allows us to write $w = Gv$ for $G \in \mathcal{H}_\infty$ , $v \in \mathcal{L}_{2e}[0,\infty)$ (or $G \in \mathcal{H}_\infty^K, v \in \mathcal{L}_{2e}^K[0,\infty)$) to mean that $w$ satisfies

$$P_T w = P_T G P_T v\ \forall 0 \le T < \infty.$$

Thus $G \in \mathcal{H}_\infty$ is bounded not only on $\mathcal{L}_2[0,\infty)$ but also on $\mathcal{L}_{2e}[0,\infty)$ [3]. It is easy to show that $G \in \mathcal{H}_\infty^K$ if and only

[2]A cone $K$ is said to be proper if it is convex ($x, y \in K; \alpha, \beta \ge 0 \Rightarrow \alpha x + \beta y \in K$), pointed ($K \cap (-K) = \{0\}$), closed and solid ($intK \neq \phi$).

[3]Boundedness is understood in an extended sense, as the boundedness of every restriction $P_T\mathcal{L}_2 \to P_T\mathcal{L}_2$, whose induced norms are uniformly bounded with respect to $T$.

if its causal extension satisfies $G\mathcal{L}_{2e}^K[0,\infty) \subseteq \mathcal{L}_{2e}^K[0,\infty)$. Finally, it is elementary to prove that $\hat{x}(0) \in K$ is necessary if $x \in \mathcal{L}_{2e}^K[0,\infty)$, and that $\hat{G}(0)K \subseteq K$ if $G \in \mathcal{H}_\infty^K$.

The spectrum of a square matrix $M$ and related quantities are defined by

$$\begin{aligned} spec(M) &:= \{\lambda \in \mathbb{C} : \lambda I - M \text{ is a singular matrix.}\} \\ \rho(M) &:= \max\{|\lambda| : \lambda \in spec(M)\} \quad \text{(Spectral radius)} \\ \mu(M) &:= \max\{Re\ \lambda : \lambda \in spec(M)\} \quad \text{(Spectral abscissa).} \end{aligned}$$

Similarly, we also define the spectrum and the spectral radius for operators in $\mathcal{H}_\infty$ by

$$\begin{aligned} spec(G) &:= \{\lambda \in \mathbb{C} : \lambda I - G \text{ is not invertible in } \mathcal{H}_\infty.\} \\ \rho(G) &:= \max\{|\lambda| : \lambda \in spec(G)\}. \end{aligned}$$

It can be shown that a bounded linear operator on a Banach space has a non-empty, closed, and bounded spectrum ([55] Theorem 1.2.11), which allows us to use $\max$ rather than $\sup$ in the above definition. The spectrum of the operator $G \in \mathcal{H}_\infty$ can also be written as

$$\begin{aligned} spec(G) &= \left\{\lambda \in \mathbb{C} : \text{A matrix valued function } \lambda I - \hat{G}(s) \text{ is not invertible in } \mathcal{H}_\infty.\right\} \\ &= \left\{\lambda \in \mathbb{C} : \text{There exists } s \in \bar{\mathbb{C}}_+ \text{ such that } \lambda I - \hat{G}(s) \text{ is a singular matrix. }\right\} \\ &= \bigcup_{s \in \bar{\mathbb{C}}_+} \left\{\lambda \in \mathbb{C} : \lambda I - \hat{G}(s) \text{ is a singular matrix. }\right\} \\ &= \bigcup_{s \in \bar{\mathbb{C}}_+} spec(\hat{G}(s)). \end{aligned} \tag{3.1}$$

## 3.3 Monotone Dynamical Systems

The notion of monotonicity (see [50, 49] and references therein) can be defined on a wide class of nonlinear systems. Although our focus in this chapter is input-output behaviors of linear monotone systems, their internal descriptions (state space models) are often useful to understand their properties.

Assume that we are given a proper cone $K \subset \mathbb{R}^n$, which endows $\mathbb{R}^n$ with a partial ordering $\geq_K$. We will drop subscripts from inequality signs and simply write $\geq$ when there is no danger of confusion. Let us consider a dynamical system of the form

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0. \tag{3.2}$$

Here, we assume that the function $f : \Omega \to \mathbb{R}^n$ is Lipschitz continuous on some open subset $\Omega$ containing $K$, and that a global solution $x(\cdot, x_0, f) : \mathbb{R}_+ \to \mathbb{R}^n$ exists. A dynamical system (3.2) is *monotone* (with respect to $K$) if

$$x_1 \geq x_2 \Rightarrow x(t, x_1, f) \geq x(t, x_2, f) \forall t \geq 0. \tag{3.3}$$

Let $K^*$ denote the dual cone of $K$. A function $f$ is said to be *quasimonotone* (see e.g., [49]) if

$$x_1 \geq x_2, \langle \eta, x_1 \rangle = \langle \eta, x_2 \rangle, \eta \in K^* \Rightarrow \langle \eta, f(x_1) \rangle \geq \langle \eta, f(x_2) \rangle.$$

Intuitively, this condition means that the vector field flows inward on the boundary of $K$. It is known that (3.2) is monotone if and only if $f$ is quasimonotone. If $f$ is a linear function of the form of $f(x) = Ax$, the quasimonotonicity of $f$ is equivalent to the *cross-positivity* (defined in [56]) of the matrix $A$ on $K$ [4]. The following Lemma is another generalization of the Perron-Frobenius Theorem.

**Lemma 1** *([56], Theorem 6) Let $K \subset \mathbb{R}^n$ be a proper cone and let $A$ be cross positive on $K$. Then $\mu(A)$ is an eigenvalue of $A$ and a corresponding eigenvector lies in $K$.*

Quasimonotonicity also allows us to compare solutions of two ordinary differential equations.

**Lemma 2** *(Comparison Principle[5]) Let $f, g : \Omega \to \mathbb{R}^n$ be Lipschitz continuous and assume that there exist global solutions $x(\cdot, x_1, f), x(\cdot, x_2, g) : \mathbb{R}_+ \to \mathbb{R}^n$. Assume that either $f$ or $g$ is quasimonotone and that $f(x) \geq g(x)$ for every $x \in K$. Then*

$$x_1 \geq x_2 \Rightarrow x(t, x_1, f) \geq x(t, x_2, g)\ \forall t \geq 0.$$

The definition of monotone systems can be naturally extended to controlled systems [50]. Assume that proper cones $K_x \subset \mathbb{R}^n, K_u \subset \mathbb{R}^m$ are given, which endows $\mathbb{R}^n$ and $\mathbb{R}^m$ with partial orderings. Suppose that a function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ and the control input $u : \mathbb{R}_+ \to \mathbb{R}^m$ satisfy appropriate conditions so that there exists a unique global solution $x(\cdot, x_0, u, f) : \mathbb{R}_+ \to \mathbb{R}^n$ to the differential equation

$$\dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0. \tag{3.4}$$

[4] A square matrix $A$ is said to be cross-positive on $K$ if $\xi \in K, \eta \in K^*, \langle \eta, \xi \rangle = 0 \Rightarrow \langle \eta, A\xi \rangle \geq 0$

[5] More general form can be found in [49], Theorem 4.1. See also [57], Theorem 2.3 for the case of $K = \mathbb{R}^n_+$.

Then, a dynamical system (3.4) is said to be *monotone* (with respect to $K_x$ and $K_u$) if

$$x_1 \geq x_2 \text{ and } u_1(t) \geq u_2(t)\ \forall t \geq 0 \Rightarrow x(t, x_1, u_1, f) \geq x(t, x_2, u_2, f) \forall t \geq 0. \tag{3.5}$$

## 3.4 Properties of Cone-preserving Transfer Functions

The purpose of this section is to show the DC-dominant property of cone-preserving transfer functions (Theorem 3). This result will be used to prove the small gain theorem for cone-preserving systems in Section 3.5. We start with the following simple observation.

**Lemma 3** *$\mathcal{H}_\infty^K$ is complete.*

**Proof:** We prove that $\mathcal{H}_\infty^K$ is closed in $\mathcal{H}_\infty$ when $K \subset \mathbb{R}^n$ is closed. Then the completeness of $\mathcal{H}_\infty^K$ follows since $\mathcal{H}_\infty$ is complete and a closed subset of a complete metric space is complete. We prove this by showing $\mathcal{H}_\infty \setminus \mathcal{H}_\infty^K$ is open. Suppose $Q_0 \in \mathcal{H}_\infty \setminus \mathcal{H}_\infty^K$. Then there exists $u_0 \in \mathcal{L}_2^K[0,\infty)$, $\|u_0\| = 1$ such that $v_0 = Q_0 u_0 \in \mathcal{L}_2[0,\infty) \setminus \mathcal{L}_2^K[0,\infty)$. This implies that there exists a set $I_0 \subset [0,\infty)$ with positive Lebesgue measure such that $d(t) > 0\ \forall t \in I_0$, where $d(t)$ is the distance between $v_0(t)$ and $K$ defined by

$$d(t) = \inf\{\|p - v_0(t)\| : p \in K\}.$$

Hence $\sqrt{\int_{I_0} d(t)^2 dt} = \epsilon > 0$. If $Q \in \mathcal{H}_\infty^K$, then $v = Qu_0 \in \mathcal{L}_2^K[0,\infty)$ and

$$\begin{aligned}
\|Q - Q_0\| &= \sup_{u \in \mathcal{L}_2, \|u\|=1} \|(Q - Q_0)u\| \\
&\geq \|(Q - Q_0)u_0\| \\
&= \|v - v_0\| \\
&= \sqrt{\int_{[0,\infty)} \|v - v_0\|^2 dt} \\
&\geq \sqrt{\int_{I_0} \|v - v_0\|^2 dt} \\
&\geq \sqrt{\int_{I_0} d(t)^2 dt} = \epsilon.
\end{aligned}$$

The last inequality follows because $v(t) \in K$ and

$$\|v(t) - v_0(t)\| \geq d(t)\ \forall t \in I_0.$$

This implies that if $\|Q - Q_0\| < \epsilon$, then $Q \in \mathcal{H}_\infty \setminus \mathcal{H}_\infty^K$. Thus $\mathcal{H}_\infty \setminus \mathcal{H}_\infty^K$ is open. ■

Next, we show that if $G \in \mathcal{H}_\infty$, then the supremum of $\rho(\hat{G}(s))$ over $\bar{\mathbb{C}}_+$ can be attained only by an $s$ on the imaginary axis. Although this fact has an explicit connection to the generalized Nyquist criterion and the spectral radius stability criterion (Theorem 4.7 & 4.9 in [58]), we provide a proof below for the sake of completeness.

**Lemma 4** *Let $G \in \mathcal{H}_\infty$ and $r := \sup_{s\in\bar{\mathbb{C}}_+} \rho(\hat{G}(s))$. Then there does not exist $s \in \mathbb{C}_+$ such that $r = \rho(\hat{G}(s))$.*

**Proof:** Suppose there exists $s^* \in \mathbb{C}_+$ such that $r = \rho(\hat{G}(s^*))$. Then, by definition of the spectral radius, there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = r$ and $det(\lambda I - \hat{G}(s^*)) = 0$. Hence $s = s^*$ is a zero of an analytic function

$$f(s) = det(I - \frac{1}{\lambda}\hat{G}(s))$$

defined on $\bar{\mathbb{C}}_+$. Since $f(s)$ is analytic and non-constant, it has distinct zeros ([59], p.79). Now, let us consider the Nyquist contour depicted in Figure 3.1 and parametrized by $\gamma$ and its image parametrized by $f \circ \gamma$. If $n(\sigma, p)$ denotes the winding number of the curve $\sigma$ around $p$, by the argument principle ([59], p.123),

$$n(f \circ \gamma, 0) = \frac{1}{2\pi j}\int_{f\circ\gamma} \frac{dz}{z} = \frac{1}{2\pi j}\int_{\gamma} \frac{f'(s)}{f(s)} ds = \sum_{i=1}^{m} n(\gamma, s_i)$$

where $s_1, \cdots, s_m$ are zeros of $f(s)$ counted with multiplicity. Since $s^*$ is a zero of $f(s)$, $s^* \in \{s_1, \cdots, s_m\}$. Moreover, $n(\gamma, s^*) = 1$ and $n(\gamma, s_i)$ is either $0$ or $1$ for each $s_i$. Hence $n(f \circ \gamma, 0) \geq 1$, which implies that the curve $f \circ \gamma$ encircles the origin at least once. For $t \in [0, 1]$, consider the following 1-parameter family of maps, obtained from homotopy from $f$:

$$\sigma_t = det(I - \frac{t}{\lambda}\hat{G} \circ \gamma).$$

Notice that $\sigma_1 = f \circ \gamma$ and $\sigma_0 \equiv 1$. Since $n(\sigma_1, 0) \geq 1$ and $n(\sigma_0, 0) = 0$, there exists $t \in (0, 1)$ such that $0 \in \sigma_t$. This implies that

$$det(I - \frac{t}{\lambda}\hat{G}(s)) = 0$$

for some $s \in \gamma \subset \bar{\mathbb{C}}_+$ and $t \in (0, 1)$. This further implies that $det(\frac{\lambda}{t} I - \hat{G}(s)) = 0$ for some $s \in \bar{\mathbb{C}}_+$, meaning that

$$\frac{\lambda}{t} \in spec(\hat{G}(s)).$$

Therefore, the supremum of $\rho(\hat{G}(s))$ over $\bar{\mathbb{C}}_+$ is at least $|\lambda/t| > |\lambda| = r$. However, this contradicts the definition of $r$. ■

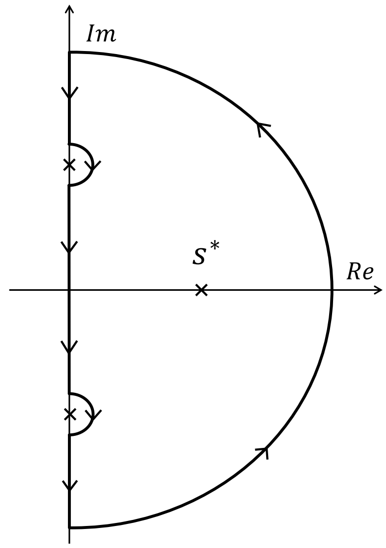


Figure 3.1: The Nyquist contour includes the imaginary axis and an infinite semi-circle into the right half plane, and the path $\gamma$ goes anti-clockwise around the Nyquist contour. When $f(s)$ has zeros on the imaginary axis, $\gamma$ avoids them by making infinitesimal semi-circles around them into the right half plane.

The next Lemma shows that the spectral radius $\rho(\hat{G}(j\omega))$ attains its maximum at $\omega = 0$ when $G$ is cone preserving.

**Lemma 5** *Let $K \subseteq \mathbb{R}^n$ be a proper cone. If $G \in \mathcal{H}_\infty^K$, then $\rho(\hat{G}(j\omega)) \leq \rho(\hat{G}(0))$ $\forall \omega \in \mathbb{R}$.*

**Proof:** Suppose on the contrary that $\rho(\hat{G}(j\omega_0)) > \rho(\hat{G}(0))$ for some nonzero $\omega_0$. Our strategy of proof is to construct a signal $u \in \mathcal{L}_{2e}^K[0,\infty)$ containing two frequency components at $\omega = 0$ and $\omega = \omega_0$ such that $w = G^N u \notin \mathcal{L}_{2e}^K[0,\infty)$ for some positive integer $N$.

Let $\lambda_{\omega_0}$ be the eigenvalue of $\hat{G}(j\omega_0)$ such that $|\lambda_{\omega_0}| = \rho(\hat{G}(i\omega_0))$ and $\xi$ be the corresponding eigenvector, i.e.,

$$\hat{G}(j\omega_0)\xi = \lambda_{\omega_0}\xi, \ \|\xi\| = 1 \tag{3.6}$$

Construct $v \in \mathcal{L}_{2e}[0,\infty)$ by

$$v(t) = \begin{bmatrix} |\xi_1|\sin(\omega_0 t + \alpha_1) \\ \vdots \\ |\xi_n|\sin(\omega_0 t + \alpha_n) \end{bmatrix}, \ \alpha_i = \angle\xi_i.$$

and let $V = \{x \in \mathbb{R}^n : \exists t \geq 0 \text{ s.t. } v(t) = x\}$ be its trajectory.

Since $K$ has non-empty interior, the set

$$\mathcal{D} = \{(c,p) \in (\mathbb{R}, K) : \|p\| = 1, cp + V \subseteq K\}$$

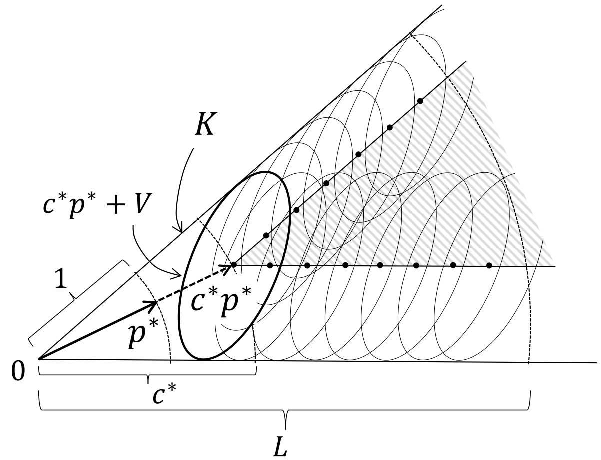


Figure 3.2: If a unit vector $p \in K$ and a constant $c$ are such that the scaled vector $cp$ belongs to the shaded area, then $(c, p) \in \mathcal{D}$. The figure also shows a graphical interpretation of $c^*$ and $p^*$.

is nonempty [6]. Also, it is easy to show that $\mathcal{D}$ is closed in the standard topology of $\mathbb{R} \times \mathbb{R}^n$. Using a sufficiently large $L$, define

$$\begin{aligned}(c^*, p^*) = \ \arg \min_{(c,p)\in(\mathbb{R},K)} \ & c \\ \text{s.t.} \quad & (c, p) \in \mathcal{D} \\ & 0 \leq c \leq L.\end{aligned}$$

Notice that the above "min" is attained since the constraint domain is compact. Moreover, $0 < c^* < \infty$ since $(0, p) \notin \mathcal{D}$ for any $p \in K$ [7].

Since $\rho(\hat{G}(j\omega_0)) > \rho(\hat{G}(0))$, by Gelfand's formula, there exists $N \in \mathbb{N}$ such that

$$\|\hat{G}(0)^N\| < \rho(\hat{G}(j\omega_0))^N.$$

Using $p^*$ defined above, define

$$r = \hat{G}(0)^N p^* \in K.$$

Notice that

$$\|r\| \leq \|\hat{G}(0)^N\| \|p^*\| < \rho(\hat{G}(j\omega_0))^N. \tag{3.7}$$

[6] Pick $p \in int(K)$ with $\|p\| = 1$ and $B(p, \epsilon) \subseteq K$. Then for all $c \geq 1/\epsilon$, $(c, p) \in \mathcal{D}$ since $V \subseteq B(0, 1)$.
[7] Since $K$ is pointed, $V \not\subseteq K$.

Now consider a periodic signal $u$ defined by

$$u(t) = c^* p^* + v(t),\ t \geq 0.$$

and $w := G^N u$. By definition of $(c^*, p^*)$, $u \in \mathcal{L}_{2e}^K[0, \infty)$. Since $G^N$ is a stable transfer function, the initial transient behavior diminishes as $t \to \infty$. From (3.6), $w(t)$ approaches a periodic signal $\tilde{\omega}(t)$ in that $\|w(t) - \tilde{w}(t)\| \to 0$ as $t \to \infty$, where

$$\begin{aligned}\tilde{w}(t) &= c^* r + |\lambda_{\omega_0}|^N v(t-\tau) \\ &= |\lambda_{\omega_0}|^N (\bar{c}\bar{p} + v(t-\tau))\end{aligned} \tag{3.8}$$

where $\bar{c} = c^*\|r\|/|\lambda_{\omega_0}|^N$, $\bar{p} = r/\|r\|$, and $\tau$ denotes a phase shift given by $\tau\omega_0 = \angle\lambda_{\omega_0}$. In the above expression, $\bar{p} \in K$ and $\|\bar{p}\| = 1$. Moreover, from (3.7), its coefficient satisfies

$$\bar{c} = \frac{c^*\|r\|}{|\lambda_\omega|^N} = \frac{c^*\|r\|}{\rho(\hat{G}(j\omega_0))^N} < c^*.$$

Therefore, by definition of $c^*$, $\bar{c}\bar{p} + V \not\subseteq K$. Since $v(t-\tau)$ follows the trajectory $V$, (3.8) means that $\tilde{w}(t_0) \notin K$ for some $t_0 \geq 0$. Let $d = \min\{\|\tilde{w}(t_0) - p\| : p \in K\} > 0$ be the distance between $\tilde{w}(t_0)$ and $K$. Since $\tilde{w}(t)$ is periodic, $\tilde{w}(t_k) \notin K$ and $\min\{\|\tilde{w}(t_k) - p\| : p \in K\} = d$ for all $t_k = t_0 + 2\pi k/\omega, k \in \mathbb{N}$.

Since $\|w(t_k) - \tilde{w}(t_k)\| \to 0$ as $k \to \infty$, there exists $t_k \geq 0$ such that $\|w(t_k) - \tilde{w}(t_k)\| < d$, meaning that $w(t_k) \notin K$. By continuity of $w(t)$ and closedness of $K$, $w(t) \notin K$ on $t \in [t_k - \epsilon, t_k + \epsilon]$. Hence we conclude that $w \notin \mathcal{L}_{2e}^K[0, \infty)$. Thus we have constructed $u \in \mathcal{L}_{2e}^K[0, \infty)$, $G^N \in \mathcal{H}_\infty^K$, $w = G^N u \notin \mathcal{L}_{2e}^K[0, \infty)$, which is a contradiction. ■

**Theorem 3** *(DC-Dominant Property) Suppose $K$ is a proper cone and $G \in \mathcal{H}_\infty^K$. Then*

$$\rho(G) = \sup_{s \in \bar{\mathbb{C}}_+} \rho(\hat{G}(s)) = \rho(\hat{G}(0)).$$

**Proof:** Noticing (3.1),

$$\begin{aligned}
\rho(G) &= \max\{|\lambda| : \lambda \in spec(G)\} \\
&= \max\left\{|\lambda| : \lambda \in \bigcup_{s\in\bar{\mathbb{C}}_+} spec(\hat{G}(s))\right\} \\
&= \sup_{s\in\bar{\mathbb{C}}_+} \max\left\{|\lambda| : \lambda \in spec(\hat{G}(s))\right\} \\
&= \sup_{s\in\bar{\mathbb{C}}_+} \rho(\hat{G}(s)) \\
&= \rho(\hat{G}(0)).
\end{aligned}$$

The last equality follows from Lemma 4 and Lemma 5. ■

## 3.5 Feedback Interconnection of Cone-preserving Transfer Functions

Stability of a dynamical system is often studied by considering the feedback interconnection of transfer functions. The central tool in this approach is the small gain theorem. In this section, we introduce a refined version of the small gain theorem specialized to feedback interconnections of cone-preserving transfer functions which uses the DC-dominant property uncovered in Section 3.4. The theorem below is "more informative" than the usual small gain theorem due to the following peculiar aspects:

- If $G$ is cone-preserving, the stability of the positive feedback system in Figure 3.3 is guaranteed solely by a DC-gain condition $\rho(\hat{G}(0)) < 1$. This is in clear contrast with the case of general linear systems for which $\rho(\hat{G}(j\omega)) < 1\ \forall\omega$ is required for the stability (Theorem 4.9 in [58]).
- For $G \in \mathcal{H}_\infty$, $\rho(G) < 1$ is only a sufficient condition for invertibility of $I - G$ in $\mathcal{H}_\infty$. For example, Figure 3.3 with $G(s) = -k/(s+\epsilon)$ is stable for arbitrarily large $k > 0$ since the phase of a transfer function $k/(s+\epsilon)$ never reaches $-180°$. In contrast, if $G \in \mathcal{H}_\infty^K$, the condition $\rho(G) < 1$ is in fact *necessary and sufficient* for invertibility of $I - G$ in $\mathcal{H}_\infty^K$.
- If $G$ is cone-preserving and its spectral radius is less than one, then $(I-G)^{-1}$ is again a cone-preserving transfer function. In particular, this implies that $r \mapsto y$ in Figure 3.3 is again a cone-preserving map. Namely, a positive feedback loop does not destroy the cone-preserving property.

**Theorem 4** *(Small Gain Theorem for cone preserving systems) Let $K \subset \mathbb{R}^n$ be a proper cone and $G \in \mathcal{H}_\infty^K$. Then the following statements hold.*

*(I) If $\rho(\hat{G}(0)) < 1$, then $(I-G)$ is invertible in $\mathcal{H}_\infty^K$, and $(I-G)^{-1} = \sum_{k=0}^{\infty} G^k$.*

*(II) If $\rho(\hat{G}(0)) = 1$, then $(I-G)$ is not invertible in $\mathcal{H}_\infty$.*

*(III) If $\rho(\hat{G}(0)) > 1$, then $(I-G)$ may or may not be invertible in $\mathcal{H}_\infty$, but is not invertible in $\mathcal{H}_\infty^K$.*

**Proof:** (I) Define a sequence $\{T_k\}$ in $\mathcal{H}_\infty^K$ by $T_k = \sum_{i=0}^{k} G^i$. Since $\|G^k\|^{1/k} \to \rho(G)$ and $\rho(G) < 1$, there exist $0 < \delta < 1$ and $N \in \mathbb{N}$ such that $\|G^k\|^{1/k} < \delta \; \forall k > N$. For $N < n < m$,

$$\|T_m - T_n\| = \| \sum_{l=n+1}^{m} G^l \| \leq \sum_{l=n+1}^{m} \|G^l\| \leq \sum_{l=n+1}^{m} \delta^l \leq \frac{\delta^{n+1}}{1-\delta} \to 0 \text{ as } n \to \infty.$$

Hence $\{T_k\}$ is a Cauchy sequence. Since $\mathcal{H}_\infty^K$ is complete (Lemma 3), $T_\infty := \lim_{k\to\infty} T_k$ exists in $\mathcal{H}_\infty^K$. Moreover,

$$\lim_{k\to\infty} \|T_k(I-G) - I\| = \lim_{k\to\infty} \|(I-G)T_k - I\| = 0.$$

which means that $T_\infty(I-G) = (I-G)T_\infty = I$. Thus $(I-G)^{-1} = \sum_{k=0}^{\infty} G^k \in \mathcal{H}_\infty^K$ exists.

(II) By Theorem 3, $\rho(\hat{G}(0)) = \rho(G) = 1$ and $\hat{G}(0)$ is a real square matrix such that $\hat{G}(0)K \subseteq K$. Since $\hat{G}(0)$ is a compact operator on $\mathbb{R}^n$, by the Krein-Rutman theorem, $1 \in spec(\hat{G}(0)) \subset spec(G)$. By definition of $spec(G)$, $I-G$ is not invertible in $\mathcal{H}_\infty$.

(III) It suffices to show that if $I-G$ is invertible in $\mathcal{H}_\infty$, then it is not $K$-preserving. By Theorem 3, $\rho(\hat{G}(0)) = \rho(G) > 1$. The Krein-Rutman theorem implies that there exists a nonzero vector $x \in K$ such that $\hat{G}(0)x = \rho(G)x$. Define a constant signal $u(t) \equiv x$, whose Laplace transform is $\hat{u}(s) = \frac{1}{s}x$. Suppose $(I - \hat{G}(s))^{-1} \in \mathcal{H}_\infty$ and

$$\hat{v}(s) = (I - \hat{G}(s))^{-1}\hat{u}(s).$$

Then $\lim_{t\to\infty} v(t)$ exists, since the only pole of $\hat{v}(s)$ is at $s = 0$. By the final value theorem,

$$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{s\to 0} s\hat{v}(s) = \lim_{s\to 0} s(I-\hat{G}(s))^{-1}\frac{1}{s}x \\ &= (I-\hat{G}(0))^{-1}x = \frac{1}{1-\rho(G)}x \in -K. \end{aligned}$$

Thus $I-G$ is not $K$-preserving. ■

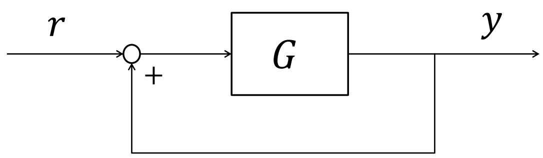


Figure 3.3: Positive Feedback

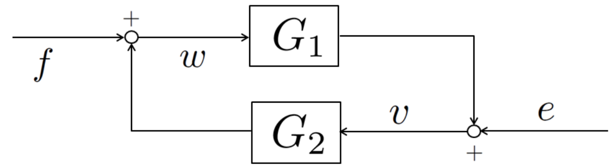


Figure 3.4: Interconnected transfer functions

It may be useful to interpret Theorem 4 as a stability theorem for feedback systems. We denote by the block diagram in Figure 3.4 the set of equations with a feedback relation

$$\hat{v}(s) = \hat{G}_1(s)\hat{w}(s) + \hat{e}(s)$$
$$\hat{w}(s) = \hat{G}_2(s)\hat{v}(s) + \hat{f}(s).$$

Internal signals $v, w \in \mathcal{L}_{2e}[0,\infty)$ are induced by input signals $e, f \in \mathcal{L}_{2e}[0,\infty)$. Physically, internal signals $v, w$ satisfying the above relationship are observed in response to the injection of signals $e, f$ when the initial state of $G_1$ and $G_2$, if they have state space models, are zero. We say that the feedback interconnection in Figure 3.4 is well-posed if the closed loop transfer function $G_{cl} : (e, f) \mapsto (v, w)$

$$G_{cl} : \mathcal{L}_{2e}[0,\infty) \times \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty) \times \mathcal{L}_{2e}[0,\infty)$$

exists. A well-posed feedback system in Figure 3.4 is said to be *stable* if $G_{cl}$ is bounded, i.e.,

$$G_{cl}(\mathcal{L}_2[0,\infty) \times \mathcal{L}_2[0,\infty)) \subseteq \mathcal{L}_2[0,\infty) \times \mathcal{L}_2[0,\infty).$$

Furthermore, a well-posed feedback system is said to be $K$*-preserving* if it satisfies

$$G_{cl}(\mathcal{L}_{2e}^K[0,\infty) \times \mathcal{L}_{2e}^K[0,\infty)) \subseteq \mathcal{L}_{2e}^K[0,\infty) \times \mathcal{L}_{2e}^K[0,\infty).$$

If $G_1$ and $G_2$ are cone-preserving transfer functions, Theorem 4 says that the feedback system in Figure 3.4 is stable and $K$-preserving if and only if $\rho(G_1 G_2) = \rho(\hat{G}_1(0)\hat{G}_2(0)) < 1$. Sufficiency can be verified, since $G_{cl}$ has an explicit form:

$$\begin{bmatrix} \hat{v}(s) \\ \hat{w}(s) \end{bmatrix} = \begin{bmatrix} (I - \hat{G}_1(s)\hat{G}_2(s))^{-1} & (I - \hat{G}_1(s)\hat{G}_2(s))^{-1}\hat{G}_1(s) \\ \hat{G}_2(s)(I - \hat{G}_1(s)\hat{G}_2(s))^{-1} & \hat{G}_2(s)(I - \hat{G}_1(s)\hat{G}_2(s))^{-1}\hat{G}_1(s) + I \end{bmatrix} \begin{bmatrix} \hat{e}(s) \\ \hat{f}(s) \end{bmatrix}.$$

in which $(I-\hat{G}_1\hat{G}_2)^{-1} \in \mathcal{H}_\infty^K$. Necessity also holds by Theorem 4 (II) and (III), since if $\rho(G_1G_2) = \rho(\hat{G}_1(0)\hat{G}_2(0)) \geq 1$, the transfer function $(I - \hat{G}_1\hat{G}_2)^{-1}$, if exists, cannot be in $\mathcal{H}_\infty^K$. As an example, consider $\mathbb{R}_+$-preserving transfer functions $\hat{G}_1 = \frac{1}{s+1}$ and $\hat{G}(s) = 1$, in which case $\rho(G_1G_2) = 1$. If $e = 0$, then $\dot{v} = f$. Hence the map $f \mapsto v$ is a pure integrator, and is not bounded on $\mathcal{L}_{2e}[0, \infty)$. Therefore, the interconnection is not stable. For another example, suppose $G_1 = 2$ and $G_2 = 1$, and $\rho(G_1G_2) = 2$. In this case, $(I-\hat{G}_1\hat{G}_2)^{-1} = -1$ belongs to $\mathcal{H}_\infty$. However, the interconnection is not $\mathbb{R}_+$-preserving, since positive signals $(e, f) = (1, 1) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ induce $(v, w) = (-3, -2) \notin \mathbb{R}_+^n \times \mathbb{R}_+^n$.

## 3.6 Examples of Cone-Preserving Systems

### 3.6.1 Positive systems

Consider a transfer function

$$\hat{G}(s) = \frac{s+2}{s^3 + 4s^2 + 4s + 1}.$$

This system is externally positive, that is, $G \in \mathcal{H}_\infty^K$ where $K = \mathbb{R}_+$, because it has an internally positive realization

$$\dot{x} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u;\ y = \begin{bmatrix} \frac{2}{3} & 0 & 0 \end{bmatrix} x.$$

Let us consider the spectrum of $G$. Noticing (3.1), it turns out that $spec(G)$ is the region surrounded by the Nyquist plot of $\hat{G}(j\omega)$. Figure 3.5 gives a graphical interpretation of the DC-dominant property of $G$. As Lemma 5 implies, we observe that $\rho(\hat{G}(j\omega))$ attains its maximum at $\omega = 0$, and as Theorem 3 implies, $\rho(G) = 2$ coincides with $\rho(\hat{G}(0))$.

### 3.6.2 Systems preserving semidefinite cones

Consider a matrix differential equation

$$\frac{d}{dt}X(t) = AX(t) + X(t)A^T + BU(t)B^T,\ X(0) = 0. \tag{3.9}$$

The above system defines a semidefinite-cone-preserving operator. To see this more precisely, let us introduce a vector representation of a square matrix $X$

$$vec(X) := \left[ \begin{array}{ccc|ccc|c|ccc} X_{11} & \cdots & X_{n1} & X_{12} & \cdots & X_{n2} & \cdots & X_{1n} & \cdots & X_{nn} \end{array} \right]^T \in \mathbb{R}^{n^2}.$$

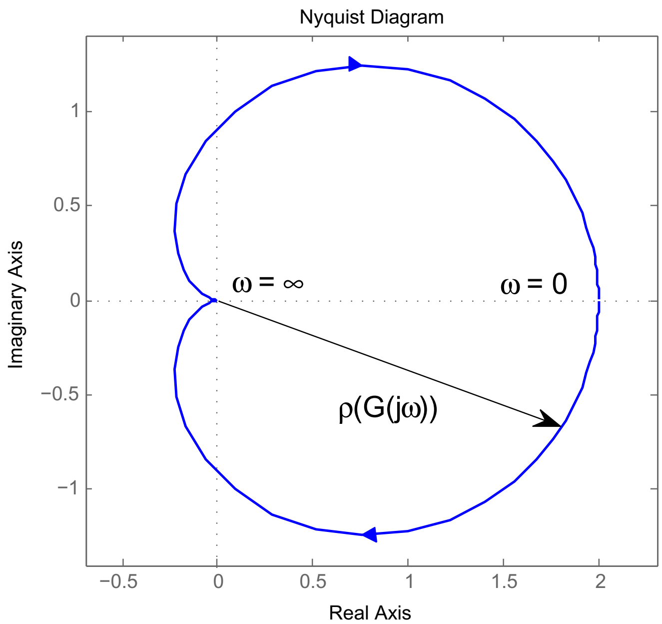


Figure 3.5: Nyquist plot of $\hat{G}(s)$

If $X$ is a symmetric matrix (i.e., $X_{ij} = X_{ji}$), then $vec(X)$ has redundant components. To remove redundancy, let us also introduce

$$\overline{vec}(X) := \left[\begin{array}{cccc|cccc|c|c} X_{11} & \sqrt{2}X_{21} & \cdots & \sqrt{2}X_{n1} & X_{22} & \sqrt{2}X_{32} & \cdots & \sqrt{2}X_{n2} & \cdots & X_{nn} \end{array}\right]^T \in \mathbb{R}^{\frac{1}{2}n(n+1)}.$$

There exists a matrix $F$ of size $n^2 \times \frac{1}{2}n(n+1)$ such that $vec(X) = F\overline{vec}(X)$ for every symmetric matrix $X$ and $F^TF = I$. Using these notations, (3.9) can be written in a vector form as

$$\frac{d}{dt}vec(X) = (A \otimes I + I \otimes A)vec(X) + (B \otimes B)vec(U),\ X(0) = 0.$$

where $\otimes$ denotes the Kronecker product. Multiplying the above equation by $F^T$ from the left, and writing

$$x(t) = \overline{vec}(X(t)),\ u(t) = \overline{vec}(U(t))$$

we have

$$\dot{x}(t) = A_0x(t) + B_0u(t),\ x(0) = 0 \tag{3.10}$$

where

$$A_0 = F^T(A \otimes I + I \otimes A)F,\ B_0 = F^T(B \otimes B)F. \tag{3.11}$$

By writing (3.9) in the form of (3.10), it is apparent that it is a $\frac{1}{2}n(n+1)$ dimensional linear system. The solution of (3.9) is given by

$$X(t) = \int_0^t e^{A(t-s)}BU(s)B^T e^{A^T(t-s)}ds,$$

which implies that $X(t)$ is positive semidefinite for $t \geq 0$ if $U(t)$ is positive semidefinite for $t \geq 0$. As a result, (3.10) defines a cone-preserving transfer function

$$G : \mathcal{L}_{2e}^K[0,\infty) \to \mathcal{L}_{2e}^K[0,\infty),\ G : u \mapsto x$$

where the proper cone $K$ is defined by

$$K = \left\{x \in \mathbb{R}^{\frac{1}{2}n(n+1)} : x = \overline{vec}(X), X \geq 0\right\}.$$

## 3.7 Delay-independent Stability of Cone-Preserving Systems

If $A_0 \in \mathbb{R}^{n\times n}$ is a Metzler matrix (all off-diagonal entries are nonnegative) and $B_0 \in \mathbb{R}^{n\times n}$ is an entry-wise nonnegative matrix, then it is known that the following delay differential equation preserves the positive orthant of $\mathbb{R}^{n\times n}$:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + B_0x(t-\tau) \\ x(t) &= x_0(t) \in K = \mathbb{R}_+^n, \quad -\tau \leq t \leq 0. \end{aligned} \tag{3.12}$$

It is also known that (3.12) is asymptotically stable if and only if its delay-free case ($\tau = 0$) is asymptotically stable [60, 61, 62]. This is a significant property of positive systems because for general linear systems, the stability is a delay-dependent property (see e.g., [63]). In this section, we use the results established so far to prove that the property of "delay-independent stability" holds not only for positive systems, but also for more general cone-preserving linear systems.

**Theorem 5** *Let $K \in \mathbb{R}^n$ be a proper cone and square matrices $A_0, B_0 \in \mathbb{R}^{n\times n}$ are such that the dynamical system*

$$\dot{x}(t) = A_0x(t) + B_0u(t),\ x(0) = x_0 \tag{3.13}$$

*is monotone with respect to $K_x = K_u = K$. Then the delay differential equation*

$$\dot{x}(t) = A_0 x(t) + B_0 x(t - \tau) \tag{3.14}$$

*with the initial condition $x(t) = x_0(t) \in K \ \forall t \in [-\tau, 0]$ is asymptotically stable for any nonnegative delay $\tau$ if its delay-free case ($\tau = 0$) is asymptotically stable and monotone.*

**Proof:** Define

$$\Phi(t) = \begin{cases} e^{A_0 t} B_0 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Then the solution of (3.14) satisfies

$$x(t) = z_0(t) + \int_\tau^t \Phi(t - s) x(s - \tau) ds \ t \geq 0 \tag{3.15}$$

where $z_0 \in \mathcal{L}_2^K[0, \infty)$ is defined by

$$z_0(t) = e^{A_0 t} x_0(0) + \int_0^\tau \Phi(t - s) x_0(s - \tau) ds.$$

Define $G_\tau \in \mathcal{H}_\infty^K$ by

$$(G_\tau u)(t) = \int_\tau^t \Phi(t - s) u(s - \tau) ds.$$

In the Laplace domain representation,

$$\hat{G}_\tau(s) = e^{-\tau s}(sI - A_0)^{-1} B_0.$$

Then (3.15) can be written as $x = z_0 + G_\tau x$. If $\rho(G_\tau) < 1$, then by Theorem 4, $(I - G_\tau)^{-1} \in \mathcal{H}_\infty^K$. Thus the solution $x$ satisfies

$$x = (I - G_\tau)^{-1} z_0 \in \mathcal{L}_2^K[0, \infty)$$

and the proof is complete.

Hence it is left to show that $\rho(G_\tau) < 1$. We establish this using the hypothesis that the delay-free case of (3.14) is asymptotically stable and monotone. One consequence of the monotonicity of (3.13) is that $B_0 x \in K \ \forall x \in K$, since otherwise there exists $\xi \in K$ such that $B_0 \xi \notin K$ and by choosing $u(t) \equiv \xi$ and $x(0) = 0$, (3.13) yields a solution such that $x(t) \notin K$ for $t \in (0, \epsilon)$. Consider a differential equation

$$\dot{x}(t) = (A_0 + \kappa B_0)x(t) \qquad (3.16)$$
$$x(t) = x_0 \in K$$

and denote its solution by $x^\kappa(t)$. By the monotonicity of (3.13), $x^0(t) \geq 0$ for $t \geq 0$ (assume $u(t) \equiv 0$ in (3.13)). When $\kappa = 1$, by the hypothesis, the system is asymptotically stable and hence $x^1(t) \to 0$ as $t \to \infty$. Notice that the right hand side of (3.16) is ordered by $\kappa$, that is,

$$0 \leq \kappa \leq 1 \Rightarrow A_0 x \leq (A_0 + \kappa B_0)x \leq (A_0 + B_0)x \ \forall x \in K.$$

Hence, by Lemma 2, we have that $0 \leq x^0(t) \leq x^\kappa(t) \leq x^1(t)$ for all $t \geq 0$. Since the solution $x^\kappa(t)$ is "sandwiched" by zero and $x^1(t)$, and $x^1(t) \to 0$, we conclude the system (3.16) is monotone and asymptotically stable for $\kappa \in [0, 1]$ and for any initial state in $K$. Since (3.16) is monotone, the matrix $A_0 + \kappa B_0$ is cross-positive on $K$. If $A_0 + \kappa B_0$ is not Hurwitz, i.e., $\mu(A_0 + \kappa B_0) \geq 0$, then by Lemma 1, there exists an initial state $x_0 \in K$ with which the solution of (3.16) does not decay, which is a contradiction. Hence $A_0 + \kappa B_0$ must be Hurwitz for $\kappa \in [0, 1]$. Hence

$$det(sI - A_0 - \kappa B_0) \neq 0 \ \forall s \in \bar{\mathbb{C}}_+ \ \forall \kappa \in [0, 1].$$

This implies that

$$\begin{aligned}
\det(\lambda I - G_0(s)) &= \det\left(\lambda I - (sI - A_0)^{-1} B_0\right) \\
&= \det \lambda (sI - A_0)^{-1} (sI - A_0 - \frac{1}{\lambda} B_0) \\
&\neq 0 \ \ \forall s \in \bar{\mathbb{C}}_+ \ \ \forall \lambda \geq 1.
\end{aligned}$$

Hence $\rho(G_0) < 1$.

Finally, since $G_\tau$ is $K$-preserving, and noticing that $\hat{G}_\tau(0) = \hat{G}_0(0) \ \forall \tau \geq 0$,

$$\rho(G_\tau) = \rho(\hat{G}_\tau(0)) = \rho(\hat{G}_0(0)) = \rho(G_0) < 1.$$

∎

## 3.8 Delay-Independent Mean-Square Stability of Geometric Brownian Motion

We next use Theorem 5 of Section 3.7 to prove the delay independence of mean-square stability of geometric Brownian motions. This extends the result of [64] which establishes this property for one dimensional geometric Brownian motions.

Let $w(t)$ be the 1-dimensional standard Brownian motion starting at zero at $t = 0$. For square matrices $A, B \in \mathbb{R}^{n\times n}$, consider an $n$-dimensional geometric Brownian motion with delay:

$$dx(t) = Ax(t) + Bx(t-\tau)dw(t),\ t \geq 0 \tag{3.17}$$

with the initial condition given by

$$x(t) = x_0(t)\ \forall t \in [-\tau, 0].$$

If $\mathcal{F}_t$ is the $\sigma$-algebra generated by $w(s); s \leq t$, then for a nonnegative delay $\tau$, the function $t \mapsto x(t-\tau)$ is $\mathcal{F}_t$-measurable and (3.17) is understood in the sense of Ito integral.

Define $X(t) := Ex(t)x^T(t)$, where $E$ means the expectation with respect to the law for the Brownian motion starting at 0.

**Lemma 6** *$X(t)$ satisfies*

$$\frac{d}{dt}X(t) = AX(t) + X(t)A^T + BX(t-\tau)B^T,\ t \geq 0 \tag{3.18}$$

*with the initial condition $X(t) = x_0(t)x_0^T(t)\ \forall t \in [-\tau, 0]$.*

**Proof:** Let $G(x(t)) = x(t)x^T(t),\ t \geq 0$. By applying Ito formula,

$$\begin{aligned} G(x(t)) =& G(x_0(0)) + \int_{-\tau}^{0} Bx_0(s)x_0^T(s)B^T ds \\ &+ \int_0^t \left(Ax(s)x^T(s) + x(s)x^T(s)A^T + B(D_\tau x)(s)(D_\tau x)^T(s)B^T\right) ds \\ &+ \int_0^t \left(x(s)(D_\tau x)^T(s)B^T + B(D_\tau x)(s)x^T(s)\right) dw(s) \end{aligned} \tag{3.19}$$

where the delay operator $D_\tau$ is defined by

$$(D_\tau x)(t) = \begin{cases} 0 & 0 \leq t < \tau \\ x(t-\tau) & \tau \leq t \end{cases}$$

for both vector-valued and matrix-valued function $x(\cdot)$. Notice that $(D_\tau x)(t)$ is $\mathcal{F}_t$-measurable. By taking the expectation of (3.19),

$$\begin{aligned} X(t) =& x_0(0)x_0^T(0) + \int_{-\tau}^{0} Bx_0(s)x_0^T(0)B^T ds \\ &+ \int_0^t \left(AX(s) + X(s)A^T + B(D_\tau X)(s)B^T\right) ds. \end{aligned} \tag{3.20}$$

Here, we used the fact that the expectation of the last term of (3.19) is zero (Theorem 3.2.1 in [65]). In differential form, (3.20) is written as

$$\frac{d}{dt}X(t) = AX(t) + X(t)A^T + BX(t-\tau)B^T,\ t \geq 0$$

with the initial condition $X(t) = x_0(t)x_0^T(t)\ \forall t \in [-\tau, 0]$ as desired. ■

A stochastic process $x(t)$ is said to be mean square stable if $\lim_{t\to\infty} E\|x(t)\|^2 = 0$. The next theorem shows that the property of mean square stability of the geometric Brownian motion defined by (3.17) is delay independent.

**Theorem 6** *A geometric Brownian motion defined by (3.17) is mean square stable for all $\tau \geq 0$ if and only if its delay-free case ($\tau = 0$) is mean square stable.*

**Proof:** Mean square stability of (3.17) is equivalent to $\lim_{t\to\infty} Ex(t)x^T(t) = 0$. Thus, by Lemma 6, it is equivalent to the asymptotic stability of the matrix system

$$\begin{aligned} &\frac{d}{dt}X(t) = AX(t) + X(t)A^T + BX(t-\tau)B^T,\ t \geq 0 \\ &X(t) = x_0(t)x_0^T(t)\ \forall t \in [-\tau, 0]. \end{aligned}$$

By employing the vector representation of the same differential equation, it can be seen that the mean square stability is further equivalent to the asymptotic stability of

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + B_0x(t-\tau) \qquad (3.21) \\ &x(t) = x_0(t) \in K\ \forall t \in [-\tau, 0]. \end{aligned}$$

where $A_0$ and $B_0$ are defined in (3.11). By Theorem 5, the above delay matrix differential equation is asymptotically stable for any nonnegative delay $\tau$ if and only if its delay-free case ($\tau = 0$) is asymptotically stable and monotone. Since the monotonicity at $\tau = 0$ clearly holds, this implies that the geometric Brownian motion (3.17) is mean square stable for all nonnegative delay $\tau$ if and only if its delay-free case ($\tau = 0$) is mean square stable. ■

## 3.9 Conclusion

In this chapter, a class of square transfer functions is considered that leave a proper cone in $\mathcal{L}_2$ signals invariant. We have shown that transfer functions in this class have the attractive *DC-dominance* which, to the best of our knowledge, has not been explicitly investigated in the system & control literature. In short, this can be understood as an interpretation of the celebrated Perron-Frobenius theorem and Krein-Rutman theorem in the context of dynamical systems. More precisely, we showed the following assertion: *If the transfer functions of interest are cone-preserving, the radius of their operator spectrum is attained by DC input signals and thus, the dynamic stability of their interconnection is guaranteed solely by the static gain analysis.* Using the DC-dominant property, we have observed that the stability of cone-preserving systems is delay-independent, which was then applied to prove that the mean-square stability of a certain class of geometric Brownian motion is delay-independent.

# Chapter 4

# Positive Systems

In the previous chapter, we considered a class of linear dynamical systems that have the “cone preserving” property. In particular, we saw that a MIMO transfer function $\hat{G}(s)$ that leaves a proper cone $K$ in the space of input-output signals invariant have the DC-dominant property – the spectral radius $\rho(\hat{G}(s))$ attains its maximum over the closed right half plane at $s = 0$. We have also seen that the stability of a cone-preserving dynamical system is independent of the presence of delays in the right hand side of the differential equation.

In this chapter, we further restrict our attention to the so-called positive systems. Positive systems are special cases of cone-preserving systems in which the invariant proper cone $K$ is chosen to be the nonnegative orthant of the Euclidean space. The theory of positive systems has an explicit and deep connection with the theory of non-negative matrices, which originated from the work of Perron and Frobenius in the years 1907-1909. Positive system theory has been repeatedly revisited by many researchers from various disciplines. Extensive results on non-negative matrices and related dynamical system theory can be found in textbooks devoted to this topic [66, 67, 68, 53, 46, 69, 70, 71]. Many standard textbooks on linear algebra and matrix theory including [72] have a section on non-negative matrices. The theory of positive linear systems has found various applications in physical, social and computational problems. For example, it serves as a plausible model for many real world dynamical systems ranging from traffic flow [73], system biology [50], PageRank and other algorithms involving Markov chains [69], queuing systems, economical and ecological systems [70].

Since positive systems are cone-preserving systems, all properties we have derived in the previous section remain valid. However, the purpose of this chapter is to derive additional stronger and attractive properties of positive systems. The main contribution of this chapter is the consideration of the control synthesis for positive systems. As compared to the analysis aspect, control aspect of positive systems seems to be less studied. It turns out that a certain type of distributed control can be easily designed for positive systems. In particular, we make the following contributions:

- We show that positive systems have a stronger DC-dominant property than the general cone-preserving systems do. As proved in [74] for discrete time case and in [52] for SISO case, some matrix norms $\|\hat{G}(j\omega)\|_p$ with $p = 1, 2, \infty$ attain their maximum at zero frequency $\omega = 0$. Notice that this is a stronger property than the DC-dominance in term of the spectral radius, since the DC-dominance in spectral radius can be deduced from the DC-dominance in norms by a simple application of Gelfand's formula. We revisit the fact that $\|\hat{G}(j\omega)\|_p$ with $p = 1, 2, \infty$ attains its maximum at $\omega = 0$ for MIMO positive transfer functions, and show that this property does not necessarily hold for more general cone-preserving systems by presenting a counterexample.

- The KYP lemma for positive systems is derived. As we saw in Chapter 2, the KYP lemma plays a central role in the well-posedness analysis. In the case of positive systems, we will show that the KYP lemma can be greatly simplified. Most notably, we can assume a *diagonal* storage function without introducing conservatism in some important classes of dissipativity analysis including the bounded realness. In a sense, this can be viewed as a natural extension of a well-known *diagonal stability* property of the positive autonomous systems $\dot{x} = Ax$, which claims that the $A$ is Hurwitz stable if and only if there exists a diagonal $P > 0$ satisfying the Lyapunov inequality $A^T P + PA < 0$.

- We point out that the above observations are very useful in the control designs for positive systems. Namely, the optimal structured control synthesis [47] and the bounded control synthesis [75] in the $H_\infty$ sense can be formulated as an LP or SDP. This is a significant fact given that structured or bounded control problems are long-standing problems in linear control theory.

The reason why these strong consequences hold for positive systems but not for general cone-preserving systems is, at least in a high level discussion, that the positivity is a coordinate dependent notion, while the cone-preserving property could be with respect to any cones. This seems to make a difference in the availability of a diagonal Lyapunov function and a structured control synthesis, since both "diagonal" and "structured" are coordinate dependent notion.

In what follows, the inequality signs $<$ and $\leq$ between elements in $\mathbb{R}^n$ are understood to be the partial ordering generated by a proper cone $\mathbb{R}^n_+$. That is, $x < y \Leftrightarrow x_i < y_i \ \forall i = 1, \cdots, n$ and $x \leq y \Leftrightarrow x_i \leq y_i \ \forall i = 1, \cdots, n$ respectively.

## 4.1 Positive linear systems

A linear autonomous system $\dot{x}(t) = Ax(t), x(0) = x_0$ is said to be *positive* if $x_0 \geq 0$ implies $x(t) \geq 0 \ \forall t \geq 0$. The definition is naturally extended to the system with input and output. A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \tag{4.1a}$$

$$y(t) = Cx(t) + Du(t) \tag{4.1b}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ is said to be *internally positive* if $x_0 \geq 0, u(t) \geq 0 \ \forall t \geq 0$ implies $x(t) \geq 0, y(t) \geq 0 \ \forall t \geq 0$. Notice that these definitions are nothing but the monotonicity condition (3.3) (3.5) with $K = \mathbb{R}^n_+$. From Section 3.3 (also from [50, 49]), the positivity and the internal positivity can be expressed in the quasimonotone condition with respect to $K = \mathbb{R}^n_+$. It turns out that a linear autonomous system is positive if and only if $A$ is a Metzler matrix (all off-diagonal entries are non-negative), and (4.1) is internally positive if and only if $A$ is Metzler and $B, C, D$ are entry-wise non-negative matrices.

The linear system (4.1) is said to be *externally positive* if $x_0 = 0, u(t) \geq 0 \ \forall t \geq 0$ implies $y(t) \geq 0 \ \forall t \geq 0$. Since this is purely an input-output property, one can equivalently say that a system is externally positive if and only if its impulse response is non-negative. One theoretical challenge here is that it is NP-hard to ensure the non-negativity of the impulse response of a given rational transfer function (this is known as the Skolem-Pisot problem [76], see also [52]). Even if it is known that the transfer function is externally positive, there may or may not exist an internally positive realization[1]. However, we will not go into this direction in this thesis.

## 4.2 System norms

We start by looking at how the system norms of positive systems can be computed. Due to the DC-dominance property, drastically simpler formulas are available for positive systems than for general linear systems. We will particularly develop the $L_1$, $L_2$, and $L_\infty$ induced norm characterizations of positive systems. Our notations are standard. Vector norms on $\mathbb{C}^n$ are defined by

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \text{ for } 1 \leq p < \infty, \ \ \|x\|_\infty = \max_i |x_i|.$$

[1] A problem of finding an internally positive realization of a given externally positive transfer function is known as a positive realization problem. An example of externally positive transfer function is known for which there is no internally positive realization no matter how large the state space dimension is allowed to take [70].

Induced norms on matrices in $\mathbb{C}^{n\times m}$ are defined by

$$\|M\|_{1\to 1} = \max_j \sum_i |m_{ij}|$$
$$\|M\|_{\infty\to\infty} = \max_i \sum_j |m_{ij}|$$
$$\|M\|_{2\to 2} = \lambda_{max}(M^*M)^{1/2}$$

where $\lambda_{max}$ is the maximum eigenvalue. $\|M\|_{2\to 2}$ is the maximum singular value of $M$ and we write $\|M\|_{2\to 2} = \|M\|$ for short, to be consistent with the notations of previous chapters. Norms on Lebesgue integrable functions $f : \mathbb{R}_+ \to \mathbb{R}^n$ are defined as

$$\|f\|_{L_p} = (\int_0^\infty \|f(t)\|_p^p dt)^{1/p} \quad \text{for} \quad 1 \leq p < \infty, \quad \|f\|_{L_\infty} = ess \sup_{t\geq 0} \|f(t)\|_\infty$$

If $G : L_p \to L_p$, the induced norms of $G$ is given by

$$\|G\|_{L_p\to L_p} = \sup_{\|u\|_{L_p}\neq 0} \frac{\|Gu\|_{L_p}}{\|u\|_{L_p}}.$$

In particular, the $L_2$-induced norm is also known as the $H_\infty$ norm, and $\|G\|_{L_2\to L_2}$ will be alternatively written as $\|G\|_\infty$. Suppose that $G : L_p \to L_p$, $G : w \mapsto z$ is defined by a linear time invariant system (4.1), or $\hat{G}(s)$ in the transfer function form on the Laplace domain. Then we have that [12]

$$\|G\|_{L_1\to L_1} = \max_j \sum_i \int_0^\infty |h_{ij}(t)|dt$$
$$\|G\|_{L_\infty\to L_\infty} = \max_i \sum_j \int_0^\infty |h_{ij}(t)|dt$$
$$\|G\|_\infty = \max_{\omega\in\mathbb{R}} \|\hat{G}(j\omega)\|_{2\to 2}$$

where $h_{ij}(t)$ is the impulse response from the $j$-th input to the $i$-th output. The above formulas hold true for all linear transfer functions. However, if $G$ is externally positive, we have the following additional formulas:

**Theorem 7** *([74, 51, 52]) If G is an externally positive system. Then its $L_1$, $L_2$, and $L_\infty$ induced norm can be written* *using matrix norms of its static gain $\hat{G}(0)$ as follows:*

$$\|G\|_{L_1\to L_1} = \max_j \sum_i \int_0^\infty h_{ij}(t)dt = \|\hat{G}(0)\|_{1\to 1}$$
$$\|G\|_{L_\infty\to L_\infty} = \max_i \sum_j \int_0^\infty h_{ij}(t)dt = \|\hat{G}(0)\|_{\infty\to\infty}$$
$$\|G\|_\infty = \|\hat{G}(0)\|$$

**Proof:** The following proof of $L_1$ and $L_\infty$ norm conditions are borrowed from [74]. Since impulse responses are nonnegative signals, $|h_{ij}(t)| = h_{ij}(t)$. However, notice that

$$\int_0^\infty h_{ij}(t)dt = \int_0^\infty h_{ij}(t)e^{-st}dt|_{s=0} = \hat{G}_{ij}(0).$$

which is a non-negative quantity for every $(i, j)$. Hence

$$\|G\|_{L_1\to L_1} = \max_j \sum_i \hat{G}_{ij}(0) = \|\hat{G}(0)\|_{1\to 1}, \quad \|G\|_{L_\infty\to L_\infty} = \max_i \sum_j \hat{G}_{ij}(0) = \|\hat{G}(0)\|_{\infty\to\infty}.$$

The $H_\infty$ norm characterization was reported in [51, 52] (results are presented in discrete-time domain and the SISO case respectively). We employ an alternative approach here. To complete the proof, it is sufficient to prove that $\|\hat{G}(0)\| \geq \|\hat{G}(j\omega)\|$ for all $\omega \in \mathbb{R}$. Without loss of generality, suppose $\|\hat{G}(j\omega)\| > \|\hat{G}(0)\| = 1$ for some $\omega \neq 0$. Then there exist a set of non-zero complex vectors $\hat{w}, \hat{v}$ such that $\hat{w} = \hat{G}(j\omega)\hat{v}$ and $\|\hat{w}\| > \|\hat{v}\|$. Namely, in the steady state, the output signal $w(t)$ is induced when the input signal $v(t)$ is applied to the system, where

$$v(t) = \begin{bmatrix} |\hat{v}_1|\sin(\omega t + \alpha_1) \\ \vdots \\ |\hat{v}_n|\sin(\omega t + \alpha_n) \end{bmatrix}, \ \alpha_i = \angle\hat{v}_i \quad \text{and} \quad w(t) = \begin{bmatrix} |\hat{w}_1|\sin(\omega t + \beta_1) \\ \vdots \\ |\hat{w}_n|\sin(\omega t + \beta_n) \end{bmatrix}, \ \beta_i = \angle\hat{w}_i$$

and

$$\sum_{i=1}^n |\hat{w}_i|^2 > \sum_{i=1}^n |\hat{v}_i|^2. \tag{4.2}$$

Define the non-negative vector $v_0 = [|\hat{v}_1| \ \cdots \ |\hat{v}_n|]^T$. Then $u(t) = v_0 + v(t)$ is a non-negative signal. If the input $u(t)$ is applied to the system, the output $y(t)$ can be written as $y(t) = w_0 + w(t)$, where $w_0$ is given by $w_0 = \hat{G}(0)v_0$ and $w(t)$ is given above. By the positivity of $G$, $y(t)$ is a non-negative signal. Hence $(w_0)_i \geq |\hat{w}_i|$ for each entry.

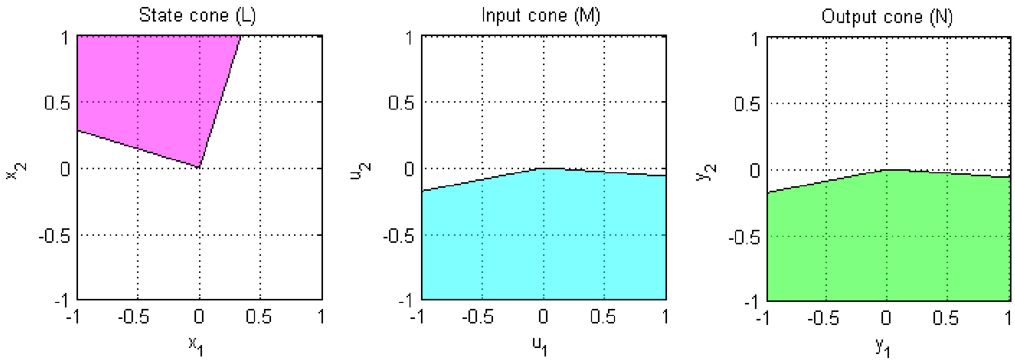


Figure 4.1: The input, state, and output cones

Combined with (4.2), we have

$$\|w_0\|^2 \geq \sum_{i=1}^{n} |\hat{w}_i|^2 > \sum_{i=1}^{n} |\hat{v}_i|^2 = \|v_0\|^2.$$

However, since $w_0 = \hat{G}(0)v_0$ this contradicts $\|\hat{G}(0)\| = 1$. ■

In Theorem 3 in Section 3.4, we have derived the DC-dominance property of a cone-preserving transfer function in terms of the spectral radius. This states that

$$\sup_{\omega \in \mathbb{R}} \rho(\hat{G}(j\omega)) = \rho(\hat{G}(0)) \tag{4.3}$$

holds for a cone-preserving transfer function $\hat{G}(s)$. Since positive systems are cone-preserving, this result certainly applies to positive systems as well. On the other hand, Theorem 7 claims the DC-dominance in terms of matrix norms. Notice that the DC-dominance in matrix norms is even stronger than the DC-dominance (4.3) in the spectral radius, since by an application of the Gelfand's formula, (4.3) can be deduced from

$$\sup_{\omega \in \mathbb{R}} \|\hat{G}(j\omega)\| = \|\hat{G}(0)\|. \tag{4.4}$$

Hence, a natural question is whether (4.4) is also true for general cone preserving transfer functions. The next counterexample shows that this is not the case. Consider the following three proper cones in $\mathbb{R}^2$, as shown in Figure 4.1, defined by

$$\mathcal{L} = \{x \in \mathbb{R}^2 : Lx \geq 0\}, \mathcal{M} = \{u \in \mathbb{R}^2 : Mu \geq 0\}, \mathcal{N} = \{y \in \mathbb{R}^2 : Ny \geq 0\}$$

where

$$L = \begin{bmatrix} -0.740 & 0.253 \\ 0.191 & 0.675 \end{bmatrix}, \quad M = N = \begin{bmatrix} 0.233 & -1.315 \\ -0.101 & -1.558 \end{bmatrix}$$

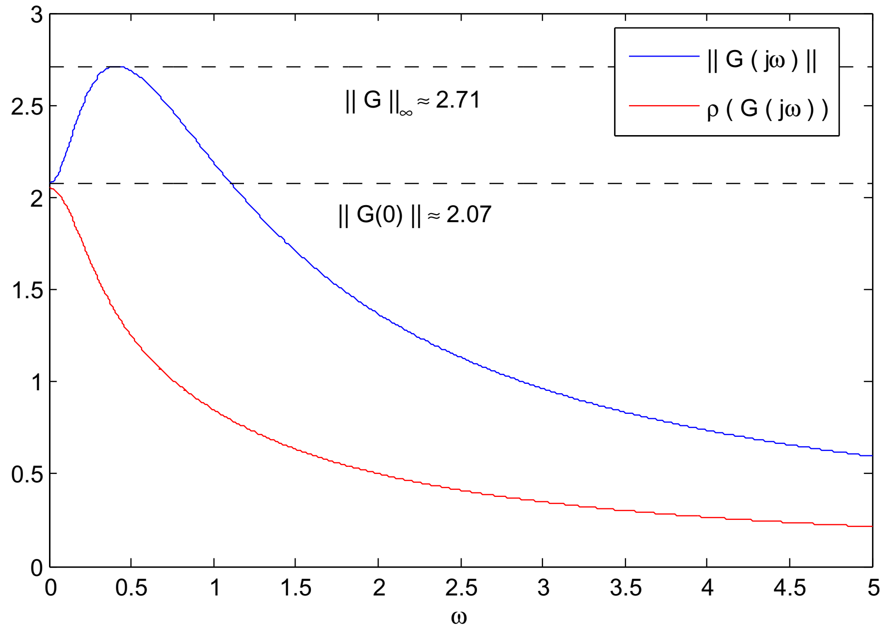


Figure 4.2: Matrix 2-norm and spectral radius of $\hat{G}(j\omega)$ when $G$ is cone-preserving but not externally positive.

and a transfer function $\hat{G}(s) = C(sI - A)^{-1}B + D$ where

$$A = \begin{bmatrix} -0.884 & 0.177 \\ 0.064 & -0.316 \end{bmatrix}, B = \begin{bmatrix} 0.191 & -1.226 \\ -0.119 & 2.285 \end{bmatrix}, C = \begin{bmatrix} 2.215 & -0.0728 \\ -0.143 & 0.281 \end{bmatrix}, D = 0.$$

Since $LAL^{-1}$ is Metzler and $LBM^{-1}, NCL^{-1}, NDM^{-1}$ are entry-wise non-negative, the transfer function defines a $(\mathcal{L}, \mathcal{M}, \mathcal{N})$-cone fractional system, using the terminology of [77]. This means that as long as $x(0) \in \mathcal{L}$ and $u(t) \in \mathcal{M}$ $\forall t \geq 0$, then $x(t) \in \mathcal{L}$ $\forall t \geq 0$ and $y(t) \in \mathcal{N}$ $\forall t \geq 0$. Since $M = N$, the transfer function is cone-preserving. The frequency dependent gain $\|\hat{G}(j\omega)\|$ is plotted in Figure 4.2. Notice that $\|\hat{G}(0)\| \approx 2.07$, while $\|\hat{G}(j\omega_0)\| \approx 2.71$ at $\omega_0 \approx 0.41$. Hence $\|\hat{G}(j\omega)\|$ does *not* attain its maximum at $\omega = 0$. However, the spectral radii are computed as $\rho(\hat{G}(0)) \approx 2.05$, $\rho(\hat{G}(j\omega_0)) \approx 1.37$. Hence, as plotted in Figure 4.2, the DC-dominance property still holds in term of the spectral radius. This verifies the result of Theorem 3.

## 4.3 Computing system norms

Theorem 7 suggests that the $L_1$ and $L_\infty$ norms of positive systems are obtained simply by computing the matrix 1- and $\infty$-norms of the DC gain matrix $\hat{G}(0)$. In this section, we consider a linear programming approach to compute them. The primary purpose of doing this is for the later use in control synthesis.

**Lemma 7** *Suppose $G$ is internally positive and $A$ is Hurwitz. Then $\|G\|_{L_\infty \to L_\infty} \leq \gamma$ if and only if the LP :*

$$\lambda \in \mathbb{R}^n,\ \lambda \geq 0$$
$$A\lambda + B\mathbb{1}_m \leq 0$$
$$C\lambda + D\mathbb{1}_m \leq \gamma \mathbb{1}_p$$

*is feasible, where inequalities are entry-wise.*

**Proof:** Suppose $\|G\|_{L_\infty \to L_\infty} \leq \gamma$. By Theorem 7, $\|D - CA^{-1}B\|_{\infty\to\infty} \leq \gamma$. Since $A$ is Hurwitz and Metzler, by the inverse positivity property (e.g., [53], p.137), $-A^{-1}$ is entry-wise non-negative. Since $D - CA^{-1}B$ is entry-wise non-negative, by definition of the matrix infinity norm, this is equivalent to $(D - CA^{-1}B)\mathbb{1}_m \leq \gamma\mathbb{1}_p$. By setting $\lambda = -A^{-1}B\mathbb{1}_m \geq 0$, we have $A\lambda + B\mathbb{1}_m = 0$ and $C\lambda + D\mathbb{1}_m \leq \gamma\mathbb{1}_p$. Thus, the LP is feasible. Converse is also immediate. ∎

**Lemma 8** *Suppose $G$ is internally positive and $A$ is Hurwitz. Then $\|G\|_{L_1 \to L_1} \leq \gamma$ if and only if the LP:*

$$\lambda \in \mathbb{R}^n,\ \lambda \geq 0$$
$$A^*\lambda + C^*\mathbb{1}_p \leq 0$$
$$B^*\lambda + D^*\mathbb{1}_p \leq \gamma \mathbb{1}_m$$

*is feasible, where inequalities are entry-wise.*

**Proof:** Since $\|M\|_{1\to 1} = \|M^*\|_{\infty\to\infty}$ for any matrix $M$, the result follows from Lemma 7. ∎

## 4.4 KYP Lemma for internally positive systems

By Theorem 7, the $H_\infty$ norm of a positive system $G$ is equal to the matrix 2-norm $\|\hat{G}(0)\|$ of a static gain matrix. Alternatively, the computation of the $H_\infty$ norm can be formulated as an SDP. The following lemma (a slight variant is presented in [78]) is a key observation.

**Lemma 9** *Suppose $M_l$, $l = 1, 2, \cdots, m$ are symmetric and Metzler matrices. If $\mathbf{\Psi}_l = \{kM_l : k \geq 0\}$, then $(\mathbf{\Psi}_1, \cdots, \mathbf{\Psi}_m)$ are mutually lossless. Moreover, if there exists a nonzero matrix $X \geq 0$ such that*

$$trM_1X \geq 0, \cdots, trM_nX \geq 0$$

*then, there exists a nonzero vector $\zeta \in \mathbb{R}^m_+$ such that*

$$\zeta^* M_1 \zeta \geq 0, \cdots, \zeta^* M_m \zeta \geq 0.$$

**Proof:** Let $x_{ij}$ be the $(i,j)$-th entry of $X$, and construct $\zeta = (\sqrt{x_{11}}, \cdots, \sqrt{x_{mm}})^T \in \mathbb{R}^m_+$. Then, for each $l = 1, \cdots, m$,

$$\zeta^* M_l \zeta - tr M_l X = \sum_{i,j} (M_l)_{i,j} (\sqrt{x_{ii}}\sqrt{x_{jj}} - x_{ij}) \geq 0.$$

The inequality follows since $(M_l)_{i,j} \geq 0$ for $i \neq j$ and $X \geq 0$ implies $\sqrt{x_{ii}x_{jj}} \geq x_{ij}$. ■

The next result can be viewed as the KYP lemma for internally positive systems.

**Theorem 8** *Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz, $B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ be entry-wise non-negative, and $G(s) = C(sI - A)^{-1}B + D$. Then the following are equivalent.*

*(I) There exists a diagonal matrix $P > 0$ such that*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0$$

*(II) $\|G\|_\infty < \gamma$.*

**Proof:** By the standard Bounded Real Lemma, the implication (I)⇒(II) clearly holds. To see the converse, define

$$N_i = \begin{bmatrix} A^*E_i + E_iA & E_iB \\ B^*E_i & 0 \end{bmatrix} \text{ for } i = 1, \cdots, n, \quad M = \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - \gamma^2 I \end{bmatrix}$$

where $E_i$ is a $n \times n$ matrix whose $(i,i)$-th entry is one and other entries are zero. Notice that (I) means that there exist positive numbers $p_1, \cdots, p_n$ such that

$$\sum_{i=1}^n p_i N_i + M < 0.$$

Suppose now that the SDP (I) is infeasible. Then by the separating hyperplane theorem, there exists a non-zero $X \geq 0$ such that

$$tr(\sum_{i=1}^n p_i N_i + M)X \geq 0$$

for all positive numbers $p_1, \cdots, p_n$. In particular, this means that

$$trN_1X \geq 0, \quad \cdots, \quad trN_nX \geq 0, \quad trMX \geq 0.$$

Notice that $N_1, \cdots, N_n, M$ are Metzler. Hence, by Lemma 9, there exists a nonzero $\zeta \geq 0$ such that

$$\zeta^* N_1 \zeta \geq 0, \quad \cdots, \quad \zeta^* N_n \zeta \geq 0, \quad \zeta^* M \zeta \geq 0.$$

Writing $\zeta = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$, this means that

$$\text{All diagonal entries of } (Ax_0 + Bu_0)x_0^* \text{ are non-negative, and} \tag{4.5}$$

$$\|Cx_0 + Du_0\|^2 \geq \|u_0\|^2. \tag{4.6}$$

If $u_0 = 0$, then all diagonal entries of $Ax_0x_0^*$ are non-negative. By the Barker-Berman-Plemmons result[2], it has to be that $x_0 = 0$ and contradicts $\zeta \neq 0$. This also implies that $Cx_0 + Du_0 \neq 0$ since otherwise (4.6) means $u_0 = 0$. Define

$$\Delta = \frac{u_0(Cx_0 + Du_0)^*}{\|Cx_0 + Du_0\|^2} \in \mathbb{R}_+^{m \times p}$$

then $\|\Delta\| \leq 1$. Without loss of generality, we assume $\|D\| < 1$ because otherwise clearly $\|G\|_\infty \geq 1$. Then $(I - \Delta D)$ is invertible and $u_0 = (I - \Delta D)^{-1}\Delta C x_0$. Combined with (4.5), all diagonal entries of $(Ax_0 + Bu_0)x_0^* = \tilde{A}x_0x_0^*$ are non-negative where $\tilde{A} = A + B(I - \Delta D)^{-1}\Delta C$. Since $A$ is Metzler and $B, C, D$ have non-negative entries, and the Neumann series $(I - \Delta D)^{-1} = \sum_{k=0}^{\infty}(\Delta D)^k$, which converges, also has non-negative entries, $\tilde{A}$ is again a Metzler matrix. By another application of the Barker-Berman-Plemmons result, the fact that all diagonal entries of $\tilde{A}x_0x_0^*$ are non-negative implies that $\tilde{A}$ cannot be Hurwitz. This means that the positive feedback interconnection of $G(s)$ and $\Delta$ is not internally stable. Since $\|\Delta\| < 0$, by the small gain theorem, $\|G\|_\infty \geq 1$. ■

Recall that, without an assumption of internal positivity, the standard Bounded Real Lemma states the equivalence of $\|G\|_\infty < \gamma$ and an LMI condition similar to (I), but it is not allowed to take a diagonal $P$ in general. This means that (I) is only a sufficient condition for (II) for general systems. Theorem 8 is significant in that the internal positivity assumption makes (I) a necessary and sufficient condition for (II). Recall also the role of diagonal control-Lyapunov function in the context of distributed control mentioned in Section 2.7.2. In this context, Theorem 8 suggests that we can focus only on diagonal control-Lyapunov functions in the control design process and this will not introduce

[2] $A$ is Metzler and Hurwitz if and only if $AX$ has at least one negative diagonal entry for all non-zero $X \geq 0$. This is a direct consequence of the diagonal stability of $A$. See [79].

conservatism, provided that the closed loop system is internally positive. We will consider the structured control synthesis for positive systems in the next section. For the control design purpose, the equivalent LMI to (I):

$$\begin{bmatrix} A^*P+PA & PB & C^* \\ B^*P & -\gamma I & D^* \\ C & D & -\gamma I \end{bmatrix} < 0$$

is often more useful, which can be easily derived from (I) by taking the Schur complements.

A similar technique of proof can be used to derive a lossless $\mu$-analysis condition for internally positive systems. The following theorem claims that the robust stability of internally positive systems subject to an arbitrary number of scalar uncertainties in the unit disk can be analyzed via SDP without conservatism.

**Theorem 9** *($\mu$-analysis for internally positive systems) Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz, $B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$ be entry-wise non-negative, and $G(s) = C(sI - A)^{-1}B + D$. Then the following are equivalent.*

*(I) There exist diagonal matrices $P > 0$ and $Q > 0$ such that*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0$$

*(II) For all $\Delta \in \mathbf{\Delta}$, $I - \Delta D$ is invertible and $A + B(I - \Delta D)^{-1}\Delta C$ is Hurwitz, where*

$$\mathbf{\Delta} = \{diag(\delta_1, \cdots, \delta_m) : \delta_i \in \bar{D} \ \forall i = 1, \cdots, m\}.$$

Notice that condition (II) means that the dynamical system $G(s)$ is robustly stable when interconnected to $m$ complex scalar LTI uncertainties. It is well known in the context of $\mu$-analysis that the LMI test (I) only corresponds to computing a convex upper bound of the structured singular value $\mu$ using the diagonal scaling technique (e.g., [13]). For general stable system $G(s) = C(sI - A)^{-1}B + D$ without positivity assumption, the only situations where losslessness is known to hold is when the relation "$2s + f \leq 3$" holds, where $s$ is the number of scalar blocks and $f$ is the number of full blocks [45]. Theorem 9 is interesting because it claims that the exact $\mu$-analysis can be performed for internally positive systems with an arbitrary number of scalar uncertainty blocks using LMI. This result is significant in that it breaks the "$2s+f \leq 3$" rule mentioned above since $s$ can be arbitrary. This observation supports the fact reported in [80] that the structured stability radius of Metzler systems can be explicitly computed.

**Proof:** By the standard KYP lemma for the $\mu$-analysis, (I)$\Rightarrow$(II) is clear. We prove the converse by showing its contrapositive $\neg$(I)$\Rightarrow$ $\neg$(II). To this end, notice that $\mu_{\mathbf{\Delta}}(D) < 1$ can be assumed without loss of generality, where $\mu_{\mathbf{\Delta}}(D)$ is the structured singular value of $D$ defined by

$$\mu_{\mathbf{\Delta}}(D) = \frac{1}{\min\{|\tau| : \det(I - \tau\Delta D) = 0, \Delta \in \mathbf{\Delta}\}}$$

This is because otherwise there exists $\Delta \in \mathbf{\Delta}$ such that $I - \Delta D$ is singular, and this immediately leads to $\neg$(II). Another implication of $\mu_{\mathbf{\Delta}}(D) < 1$ is $\rho(\Delta D) < 1$ for all $\Delta \in \mathbf{\Delta}$.

Now the converse can be shown using a similar technique as in the proof of Theorem 8. Define

$$N_i = \begin{bmatrix} A^*E_i + E_iA & E_iB \\ B^*E_i & 0 \end{bmatrix} \text{ for } i = 1, \cdots, n, \quad M_j = \begin{bmatrix} C^*E_jC & C^*E_jD \\ D^*E_jC & D^*E_jD - E_j \end{bmatrix} \text{ for } j = 1, \cdots, m.$$

Then (I) means that there exists positive numbers $p_1, \cdots, p_n, q_1, \cdots, q_m$ such that

$$\sum_{i=1}^{n} p_i N_i + \sum_{j=1}^{m} q_i M_i < 0.$$

If (I) is infeasible, by the separating hyperplane theorem, there exists a non-zero $X \geq 0$ such that

$$trN_1X \geq 0, \quad \cdots, \quad trN_nX \geq 0, \quad trM_1X \geq 0, \quad \cdots, \quad trM_mX \geq 0$$

Notice that $N_1, \cdots, N_n, M_1, \cdots, M_m$ are Metzler. By Lemma 9, there exists a nonzero $\zeta \geq 0$ such that

$$\zeta^*N_1\zeta \geq 0, \quad \cdots, \quad \zeta^*N_n\zeta \geq 0, \quad \zeta^*M_1\zeta \geq 0, \quad \cdots, \quad \zeta^*M_m\zeta \geq 0.$$

Writing $\zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, this means that

All diagonal entries of $(A\xi + B\eta)\xi^*$ are non-negative, and (4.7)

$$(C\xi + D\eta)_j \geq \eta_j \quad \forall j = 1, \cdots, m. \tag{4.8}$$

Define $\Delta = diag(\delta_1, \cdots, \delta_m) \in \mathbf{\Delta}$ where each $0 \leq \delta_i \leq 1$ is set to

$$\delta_j = \begin{cases} \eta_j/(C\xi + D\eta)_j & \text{if } (C\xi + D\eta)_j \neq 0 \\ 0 & \text{if } (C\xi + D\eta)_j = 0 \end{cases}$$

so that $\eta = (I - \Delta D)^{-1}\Delta C\xi$. Note that, since $\rho(\Delta D) < 1$, the inverse exists and is entry-wise non-negative because it can be written as a convergent series

$$(I - \Delta D)^{-1} = \sum_{k=0}^{\infty} (\Delta D)^k.$$

Combined with (4.7), all diagonal entries of $(A\xi + B\eta)\xi^* = \tilde{A}\xi\xi^*$ are non-negative where $\tilde{A} = A + B(I - \Delta D)^{-1}\Delta C$. Since $\tilde{A}$ is Metzler, by the Barker-Berman-Plemmons result, this fact implies that $\tilde{A}$ is not Hurwitz. ■

## 4.5 Fixed-structure static state feedback control design

In the previous section, we have characterized systems norms of internally positive systems as LP or SDP. Using this result, we now derive control syntheses for $L_\infty$ and $L_2$ induced gain minimization. Consider a system with control input

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t),\ x(0) = 0$$
$$z(t) = Cx(t) + Du(t) + Fw(t).$$

where $A \in \mathbb{R}^n$ is a Metzler matrix and $E \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, F \in \mathbb{R}^{p\times m}$ are entry-wise non-negative matrices. We want to design a static state feedback control

$$u(t) = Kx(t)$$

which optimizes a certain performance criterion while keeping internal positivity of the closed loop system. The latter requirement can be achieved by ensuring $A + BK$ to be Metzler and $C + DK$ to be entry-wise nonnegative. We additionally require that $K$ is subject to the structural constraint $\mathcal{K}$ due to the information structure in the real world. For example, if $K \in \mathbb{R}^{2\times 2}$ but $u_1$ is to be determined without observing $x_2$, the optimal controller $K$ needs to be found among the ones with the structure

$$K = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \in \mathcal{K}.$$

### 4.5.1 $L_\infty$-induced gain minimization

Notice that the closed loop system $G_{cl}$ is given by

$$\dot{x} = (A + BK)x + Ew \tag{4.9a}$$

$$y = (C + DK)x + Fw. \tag{4.9b}$$

Our goal here is to derive an algorithm to find $K \in \mathcal{K}$ that minimizes the $L_\infty$ induced norm of $G_{cl}$, while keeping $G_{cl}$ internally positive. By a direct application of Lemma 7, this can be done by solving

$$\begin{aligned} \min \;\; & \gamma \\ s.t. \;\; & \lambda \in \mathbb{R}^n, \;\; \lambda \geq 0 \\ & (A + BK)\lambda + E\mathbb{1}_m \leq 0 \\ & (C + DK)\lambda + F\mathbb{1}_m \leq \gamma \mathbb{1}_p \\ & A + BK \text{ is Metzler} \\ & C + DK \text{ is entry-wise nonnegative} \\ & K \in \mathcal{K}. \end{aligned}$$

However, notice that both $K$ and $\lambda$ are variables in the above program. Since there is a product $K\lambda$ the above program, it is not an LP anymore. However, this difficulty can be circumvented by applying the following change of variables:

$$\lambda = P\mathbb{1}_n, KP = Z \in \mathcal{K}$$

where $P$ is a diagonal matrix. Using new variables $P$ and $Z$, the above program is equivalently written as an LP:

$$\min \ \gamma \tag{4.10a}$$

$$s.t. \ \ P > 0 : \text{diagonal} \tag{4.10b}$$

$$Z \in \mathcal{K} \tag{4.10c}$$

$$(AP + BZ)\mathbb{1}_n + E\mathbb{1}_m < 0 \tag{4.10d}$$

$$(CP + DZ)\mathbb{1}_n + F\mathbb{1}_m < \gamma\mathbb{1}_n \tag{4.10e}$$

$$AP + BZ \text{ is Metzler} \tag{4.10f}$$

$$CP + DZ \text{ is entry-wise nonnegative} \tag{4.10g}$$

$$Z \in \mathcal{K}. \tag{4.10h}$$

Using a feasible solution $P, Z$ to the above LP, the optimal controller can be reconstructed by $K = ZP^{-1} \in \mathcal{K}$. Therefore, the minimum $L_\infty$-induced gain attained by a structured static state feedback controller $K \in \mathcal{K}$ is obtained by solving the LP (4.10). In [75], an LP-based stabilizing controller design method that achieves closed loop positivity is proposed. The above technique is derived in [74], and can be viewed as a generalization of [75] for the $L_\infty$ performance optimization. A related technique is used in [81] for the optimal network routing problem.

If one requires the conservation of the flow quantity, the above LP is also subject to:

$$\sum_{i=1}^{n} (AP + BZ)_{ij} = 0 \ \forall j \in \{1, \cdots, n\}.$$

### 4.5.2 $H_\infty$ norm minimization

The Bounded Real Lemma for internally positive systems (Theorem 8) can be used to design the optimal static state feedback $H_\infty$ control that maintains the closed loop internal positivity. The use of Theorem 8 has a particular advantage over the standard Bounded Real Lemma in the distributed control design. Notably, the fact that a diagonal storage function can be assumed without loss of generality implies that the structured static state feedback control design problem can be turned into an LMI problem. This finding, together with the derivation of the "diagonal" Bounded Real Lemma (Theorem 8) is deemed to be one of the contributions of this thesis.

To see how a structured control problem can be turned into an LMI, consider again the closed loop system $G_{cl}$ given by (4.9) in which $E$ and $F$ are entry-wise nonnegative matrices. Suppose that $K \in \mathcal{K}$ is a structured static state feedback controller such that the closed loop system $G_{cl}$ is stable, internally positive, and satisfies $\|G_{cl}\|_\infty < \gamma$. By Theorem

8, this is equivalent to the existence of a diagonal matrix $P > 0$ such that

$$\begin{bmatrix} (A+BK)^*P + P(A+BK) & PE & (C+DK)^* \\ E^*P & -\gamma I & F^* \\ C+DK & F & -\gamma I \end{bmatrix} < 0. \tag{4.11}$$

By right- and left-multiplying (4.11) by $diag(P^{-1}, I, I)$ and then performing the change of variables $(P^{-1}, KP^{-1}) \mapsto (W, Z)$, we obtain

$$\begin{bmatrix} AW + WA^* + BZ + Z^*B^* & E & WC^* + Z^*D^* \\ E^* & -\gamma I & F^* \\ CW + DZ & F & -\gamma I \end{bmatrix} < 0. \tag{4.12}$$

Notice that the diagonality of $P$ plays a crucial role in this step. Namely, we have $K \in \mathcal{K} \Leftrightarrow Z \in \mathcal{K}$. Thus we have the following.

**Theorem 10** *Let $K \in \mathcal{K}$ a static state feedback controller such that the closed loop system $G_{cl}$ is stable, internally positive. The optimal $H_\infty$ performance $\gamma$ and the corresponding $H_\infty$ optimal controller can be found by solving the following SDP:*

$$\min \ \gamma \tag{4.13a}$$

$$s.t. \ \ P > 0 : \textit{diagonal} \tag{4.13b}$$

$$\begin{bmatrix} AW + WA^* + BZ + Z^*B^* & E & WC^* + Z^*D^* \\ E^* & -\gamma I & F^* \\ CW + DZ & F & -\gamma I \end{bmatrix} < 0. \tag{4.13c}$$

$$AP + BZ \ \textit{is Metzler} \tag{4.13d}$$

$$CP + DZ \ \textit{is entry-wise nonnegative} \tag{4.13e}$$

$$Z \in \mathcal{K} \tag{4.13f}$$

*The optimal controller can be reconstructed by $K = ZW^{-1}$.*

### 4.5.3 Why is closed loop positivity required?

We have derived the structured control syntheses for the $L_\infty$ and $L_2$ gain minimization for positive systems. Notice that the key requirement that we are making so that the Lemma 7 and Theorem 8 are applicable is the closed loop positivity (i.e., requirements that $A + BK$ is Metzler and $C + DK$ is entry-wise non-negative). Indeed, the optimal structured control synthesis remains very challenging if this restriction is removed. In this section, we consider why this requirement is natural in practice.

Recall that positive systems typically represent dynamics of non-negative quantities, such as concentration of substances in chemical processes and absolute temperatures. As pointed out by [75], the non-negative nature of the state has to be taken into account in the control design phase as well. If the closed loop positivity is not required, it is possible that the controller tries to drive the state space to the negative region. Since this is infeasible in real systems, the system model is self-contradicting and does not represent real systems. This will cause unexpected behavior of the closed loop system and a loss of stability or control performance when the controller is implemented in the real system.

Even if there is no such physical requirements, one might be still justified to require closed loop positivity given the computational advantages that follows. This idea is similar to requiring *superstability* even when only stability is desired physically, simply to make the control problem more tractable [82].

In other circumstances, requiring closed loop positivity might be beneficial to reduce uncertainties. To see this, consider the following situation: Suppose Figure 4.3 is a general feedback in which signals can take positive or negative values, but a nonlinear uncertainty $\Delta$ satisfies more strict sector bound condition if injected signal $v$ is restricted to the nonnegative region as in Figure 4.4. In other words, $\Delta$ is more "certain" on the positive domain. Then one can focus on the set of controllers (if exists) that artificially makes the closed loop system 4.3 nonnegative, so more strict sector bound condition on $\Delta$ can be used. To demonstrate this, let $\Delta_1, \Delta_2$ be scalar uncertainties satisfying the sector bound condition of Figure 4.4. Notice that each uncertainty block satisfies an IQC

$$\alpha^2 \|v\|^2 - \|\Delta(v)\|^2 \geq 0 \; \forall v \in \mathcal{R} \tag{4.14}$$

on the entire domain and an IQC

$$\beta^2 \|v\|^2 - \|\Delta(v)\|^2 \geq 0 \; \forall v \geq 0. \tag{4.15}$$

on the positive domain. Our objective is to design a static state feedback controller $K$ which achieves robust stability. Since the uncertainty block in Figure 4.3 has structure, we apply the $\mu$-synthesis, which can be formulated as an LMI problem in the case of static state feedback design. There are two ways to approach this design problem.

*Approach 1:* (Without positivity requirement)

The IQC (4.14) is used. A controller exists if there exists a symmetric matrix $P > 0$, a real matrix $K$ of compatible

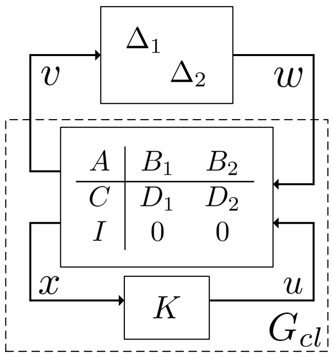


Figure 4.3: Plant description

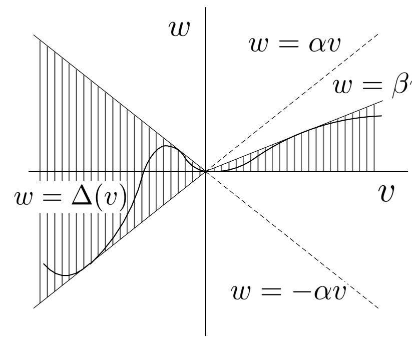


Figure 4.4: Sector bound condition

dimension, and $\Theta = diag(\theta_1, \theta_2) > 0$ such that

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_1 \\ B_1^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_1 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & -(1/\alpha^2)\Theta \end{bmatrix} \begin{bmatrix} C_{cl} & D_1 \\ 0 & I \end{bmatrix} < 0$$

where $A_{cl} = A + B_2 K$, $C_{cl} = C + D_2 K$. By standard manipulations and replacing $P^{-1}$ by $Q$, $KP^{-1}$ by $Z$, and $\alpha\Theta^{-1}$ by $\Lambda$, the condition above becomes

$$\begin{bmatrix} (AQ+B_2Z)+(AQ+B_2Z)^T & B\Lambda & (CQ+D_2Z)^T \\ \Lambda B_1^T & -(1/\alpha)\Lambda & \Lambda D_1^T \\ CQ+D_2Z & D_1\Lambda & -(1/\alpha)\Lambda \end{bmatrix} < 0 \tag{4.16}$$

Hence a robust controller exists if (4.16) admits a solution $Q > 0$, a real matrix $Z$ of a proper dimension, $\Lambda = diag(\lambda_1, \lambda_2) > 0$.

*Approach 2:* (With positivity requirement)

To use the IQC (4.15) on the positive domain, we additionally require the closed loop system $G_{cl}$ to be internally positive. There exists a controller which makes Fig. 4.3 stable and internally positive if there exists a diagonal matrix $Q > 0$, a real matrix $Z$ with a proper dimension, $\Lambda = diag(\lambda_1, \lambda_2) > 0$ satisfying

- LMI (4.16) with $\alpha$ being replaced by $\beta$
- $AQ + B_2 Z$ is Metzer
- $CQ + D_2 Z \geq\geq 0$.

Consider a second order system with

$$A = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}, B_1 = I, B_2 = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

$$C_1 = I, D_1 = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 3 & 10 \end{bmatrix}.$$

Notice that the open loop system itself is not a positive system. Suppose that the sector bound condition of the uncertainty is given by $\alpha = 0.4$, $\beta = 0.2$. A numerical analysis shows that Approach 1 is infeasible while Approach 2 is feasible. In other words, by restricting the search to the class of controllers that attains internal positivity and taking the fact that a smaller gain bound of uncertainty is available in the positive region, a stabilizing controller can be found.

Of course Approach 2 is not always feasible because there may not exist any controller achieving internal positivity of the closed loop system. Also, even if there exists such a controller, the search space of controllers in Approach 2 is smaller than in Approach 1. Hence, depending on this restriction and the description of uncertainty such as $(\alpha, \beta)$, Approach 2 may or may not provide a better result than Approach 1.

Another important advantage of Approach 2 is that a diagonal structure can be imposed on the LMI variable $Q$ without introducing conservatism. Hence one can readily synthesize a structured controller. In above numerical example, one can search for a controller gain of the form $K = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ that achieves positive stability by imposing the same structure on the LMI variable $Z$ in Approach 2. By numerical analysis it can be found feasible, yielding a controller gain $K = \begin{bmatrix} 0.500 & 0 \\ 0 & 0.675 \end{bmatrix}$.

## 4.6 Linear vs nonlinear, static vs dynamic structured controllers with positivity requirement

It is a standard knowledge in the linear $H_2$ and $H_\infty$ control theory that the best linear state feedback control law can be found in the class of linear *static* state feedback control law, if there is no additional requirements. This fact does not necessarily apply to the situation in which the controller is required to have a structure. The well-known Witsenhausen's conterexample [83] demonstrates that, although in a different context than our deterministic problems, the best control under the non-classic in formation pattern could be even nonlinear. Moreover, as in the

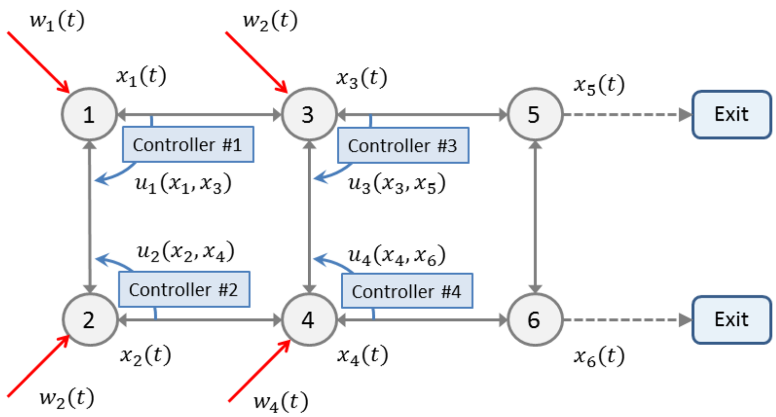


Figure 4.5: A simple traffic routing problem

second condition in Theorem 10, we additionally require that the closed loop systems is positive as well. In the previous section, we have assumed that the structured state feedback gain was static, and formulated the optimal $H_\infty$ controller synthesis within this class.

## 4.7 Application to traffic control

Let us consider a model of traffic network with six nodes shown in Figure 4.5. The system introduced here could be used as a simple model for vehicle traffic on a highway network, packet flows on a communication network, or dynamic behaviors of a flock of people under an emergency. Our goal is to find the optimal re-routing policy that reduces congestions, delays or evacuation times. It is desirable that the routing policy is implementable in a distributed manner so that each local traffic planner is able to make decisions using local information only. We demonstrate below that the structured $L_\infty$ and $H_\infty$ control synthesis presented in the previous section can be used for this purpose.

Let us denote by $x_i(t)$ the population (number of vehicles, people, etc.) at node $i$. Suppose that the dynamics of $x(t)$

satisfies the following delay differential equation.

$$\dot{x}_1(t) = -3x_1(t) + 2x_2(t-\tau) + x_3(t-\tau) + u_2(t-\tau) + w_1(t) \tag{4.17a}$$

$$\dot{x}_2(t) = 2x_1(t-\tau) - 4x_2(t) + 2x_4(t-\tau) + u_1(t-\tau) + w_2(t) \tag{4.17b}$$

$$\dot{x}_3(t) = x_1(t-\tau) - 4x_3(t) + x_4(t-\tau) + 2x_5(t-\tau) - u_1(t) + u_4(t-\tau) + w_3(t) \tag{4.17c}$$

$$\dot{x}_4(t) = 2x_2(t-\tau) + x_3(t-\tau) - 4x_4(t) + x_6(t-\tau) - u_2(t) + u_3(t-\tau) + w_4(t) \tag{4.17d}$$

$$\dot{x}_5(t) = 2x_3(t-\tau) - 3.3x_5(t) + x_6(t-\tau) - u_3(t) \tag{4.17e}$$

$$\dot{x}_6(t) = x_4(t-\tau) + x_5(t-\tau) - 2.8x_6(t) - u_4(t) \tag{4.17f}$$

In this model, assume that there are four distributed network planners (controllers) located at node 1,2,3 and 4. Functions $u_1(t), \cdots, u_4(t)$ represent the amount of traffic flow that was re-routed by controllers. For example, the first controller is able to reduce the traffic flow from node 1 to node 3 by $u_1(t)$ and send this amount of flow to node 2 instead. Notice that this operation conserves the amount of total flow, since the portion $u_1(t)$ deducted from (4.17c) is injected into (4.17b). Similarly, $u_2, u_3, u_4$ are decision variables of controller 2, 3, and 4 respectively. We further assume that each controller has access to the state variable of his own local node and an additional neighboring node. In particular, controls are determined by the following linear feedback:

$$u_1(t) = k_{11}x_1(t) + k_{13}x_3(t)$$
$$u_2(t) = k_{22}x_2(t) + k_{24}x_4(t)$$
$$u_3(t) = k_{33}x_3(t) + k_{35}x_5(t)$$
$$u_4(t) = k_{44}x_4(t) + k_{46}x_6(t).$$

For example, the first controller decides his control action $u_1(t)$ based only on $x_1(t)$ and $x_3(t)$. In the traffic model (4.17), the delay constant $\tau$ is understood to be the traveling time over each link between nodes. The closed loop system can be written as

$$\dot{x}(t) = (A_0 + B_0K)x(t) + (A_\tau + B_\tau K)x(t-\tau) + Ew(t)$$

where

$$A_0 = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.8 \end{bmatrix}, A_\tau = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, B_\tau = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & 0 & k_{13} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & k_{24} & 0 & 0 \\ 0 & 0 & k_{33} & 0 & k_{35} & 0 \\ 0 & 0 & 0 & k_{44} & 0 & k_{46} \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_6(t) \end{bmatrix}, w(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_4(t) \end{bmatrix}$$

We design two different controllers: the $L_\infty$ optimal control that minimizes the worst $L_\infty$ gain $\|x\|_\infty / \|w\|_\infty$, and the $H_\infty$ optimal control that minimizes the worst $L_2$ gain $\|x\|_2 / \|w\|_2$, while keeping internal positivity of the closed loop system. The closed loop transfer function is given by

$$\hat{G}_{K,delay}(s) = \left(sI - (A_0 + B_0 K) - (A_\tau + B_\tau K)e^{-\tau s}\right)^{-1} E. \tag{4.18}$$

In oder to guarantee internal positivity of the delay deferential equation, the internal positivity of the delay free system (4.19) is not sufficient, but the following condition [61] is required:

$A_0 + B_0 K$ is Metzler and $A_\tau + B_\tau K$ is entry-wise non-negative.

Specifically, the problem of interest is stated as follows.

**Problem 1** *Find a structured static feedback gain $K$ such that $A_0 + B_0 K$ is Metzler and $A_\tau + B_\tau K$ is entry-wise non-negative, and the closed loop system norm $\|\hat{G}_{K,delay}\|$, either $L_\infty$-induced or $L_2$-induced gain, is minimized.*

To approach this problem, let us also consider a "delay-free" closed loop transfer function obtained by setting $\tau = 0$.

$$\hat{G}_{K,delayfree}(s) = (sI - (A + BK))^{-1} E \text{ where } A = A_0 + A_\tau, B = B_0 + B_\tau \tag{4.19}$$

Using the delay-free transfer function, consider the following variant of Problem 1.

**Problem 2** *Find a structured static feedback gain $K$ such that $A_0 + B_0K$ is Metzler and $A_\tau + B_\tau K$ is entry-wise non-negative, and the closed loop system norm $\|\hat{G}_{K,delayfree}\|$, either $L_\infty$-induced or $L_2$-induced gain, is minimized.*

We claim that an optimal controller $K_2$ for Problem 2 remain optimal for Problem 1. To see this, suppose there exists a controller $K_1$ that attains closed loop internal positivity of the delayed system (4.18) and attains

$$\|\hat{G}_{K_1,delay}\| < \|\hat{G}_{K_2,delay}\| \tag{4.20}$$

where system norm can be either $L_\infty$-induced or $L_2$-induced gain. Since $\hat{G}_{K_1,delayfree}$ and $\hat{G}_{K_2,delayfree}$ define internally positive transfer functions, by the DC-dominance (Theorem 7), each side of (4.20) can be replaced by matrix norms of static gains.

$$\|\hat{G}_{K_1,delay}(0)\| < \|\hat{G}_{K_2,delay}(0)\|.$$

Since $\hat{G}_{K_1,delay}(0) = \hat{G}_{K_1,delayfree}(0)$ holds for any $K$,

$$\|\hat{G}_{K_1,delayfree}(0)\| < \|\hat{G}_{K_2,delayfree}(0)\|. \tag{4.21}$$

Since $A + BK_1$ and $A + BK_2$ are both Metzler, $\hat{G}_{K_1,delayfree}(0)$ and $\hat{G}_{K_2,delayfree}(0)$ are both internally positive. By the DC-dominance, (4.21) implies

$$\|\hat{G}_{K_1,delayfree}\| < \|\hat{G}_{K_2,delayfree}\|.$$

which contradicts the optimality of $K_2$ to Problem 2.

Using new variables in (4.10) and (4.13), the condition (4.7) is equivalent to:

$$A_0P + B_0Z \text{ is Metzler and } A_\tau P + B_\tau Z \text{ is entry-wise non-negative.} \tag{4.22}$$

Hence, to summarize, the optimal $L_\infty$ and $H_\infty$ controller for Problem 1 can be found by solving the LP and SDP (4.10) and (4.13), with conditions (4.10f) and (4.13d) being replaced by (4.22). If necessary, one can additionally require the feedback gains to be bounded as $\underline{k}_{ij} \le k_{ij} \le \bar{k}_{ij}$ for prespecified constants $\underline{k}_{ij}, \bar{k}_{ij}$ [75], by imposing $\underline{k}_{ij}P_{jj} \le Z_{ij} \le \bar{k}_{ij}P_{jj}$.

Table 4.7 summarizes the $L_\infty$ and $H_\infty$ performances of each of the $L_\infty$ and $H_\infty$ optimal controllers obtained by

|  | $L_\infty$ norm of the closed loop system | $H_\infty$ norm of the closed loop system |
|---|---|---|
| $L_\infty$ control | 3.6364 | 4.1232 |
| $H_\infty$ control | 3.7113 | 3.5660 |

Table 4.1: Closed loop performances of designed controllers.

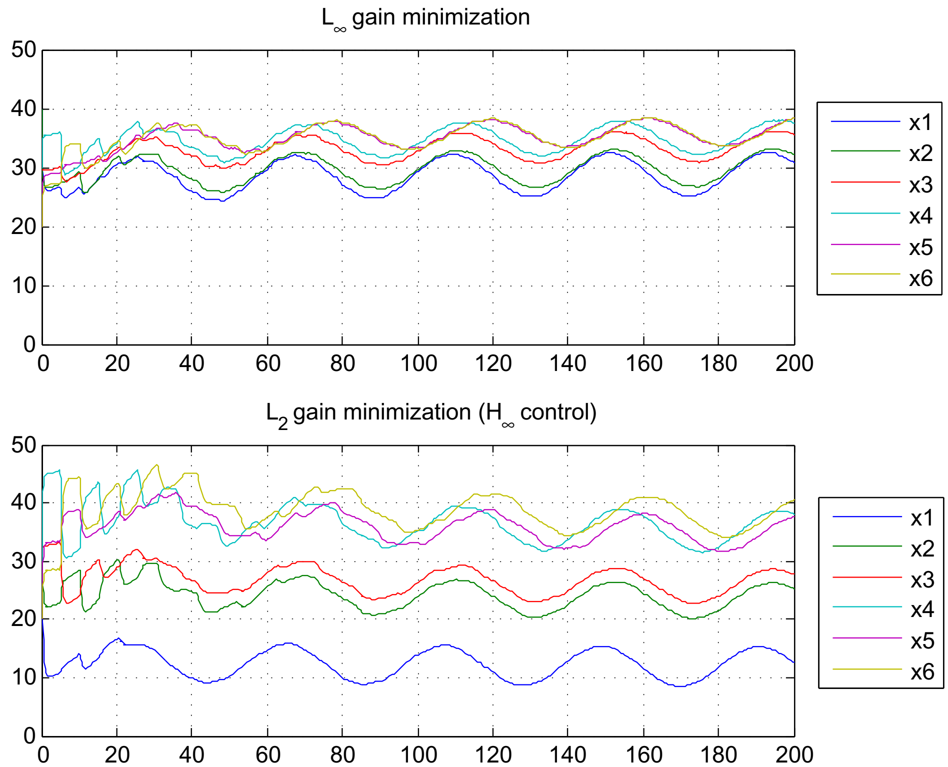


Figure 4.6: Closed loop behavior of the $L_\infty$-gain minimizing control and the $H_\infty$ control.

solving (4.10) and (4.13) for the system (4.17).

It is interesting to observe that the closed loop dynamics behave quite differently depending on the choice of controllers. Figure 4.6 shows the closed loop dynamics simulated in each case with delay constant $\tau = 5$. In each simulation, the initial state $x(0)$ and the injected signals $w_1(t), \cdots, w_4(t)$ are set to be identical. In particular, $w_1(t), \cdots, w_4(t)$ are identical non-negative signals obtained by adding a positive constant to a sinusoidal wave. As is expected from the control policy, the $L_\infty$-gain minimizing control seems to relax the congestion at the most crowded instances. The flip side is that it does not try to reduce the congestion at the less crowded instances. As a result, all nodes are made equally crowded throughout time. On the other hand, $H_\infty$ control tries to keep the entire traffic level in the network low, since by its policy, it tries to reduce the integrated squared sum of the entire population.

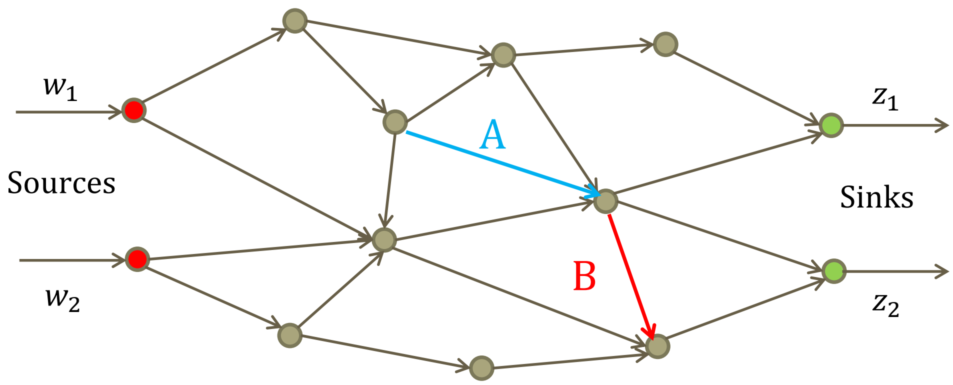


Figure 4.7: Control of traffic topology.

Another possible use of the syntheses (4.10) and (4.13) is the *topology control* of the traffic network. Sometimes it is more appropriate to consider the topology of the network as a design parameter as well, especially when it comes to the infrastructure design. In Figure 4.7, we ask which additional link (A or B) is more effective to reduce congestion after an appropriate control is implemented. By solving (4.10) and (4.13) for each situation, the optimal control policy as well as the best achievable closed loop performance are obtained. Hence a quantitative comparison between A and B is possible. Such a quantitative method for topology control might be useful not only in the traffic networks, but also in other applications such as optimal power flow designs [84]. Again, such a comparison can be carried out here because the exact value of the closed-loop gain (as opposed to a mere upper bound) is computable via LP or SDP.

## 4.8 Conclusion

In this chapter, we focused on the class of linear positive systems. We have revisit the fact that $\|G(j\omega)\|_p$ with $p = 1, 2, \infty$ attains its maximum at $\omega = 0$ for MIMO externally positive transfer functions. However, we also saw that this is a unique property of positive systems in that only $\rho(G(0)) \geq \rho(G(j\omega))\ \forall\omega \in \mathbb{R}$ hold for more general cone-preserving systems. This property is used to derive an LP/SDP formulations to compute system 1- ,2- and $\infty$-norms of internally positive systems. A major contribution of this chapter is the derivation of the KYP lemma for internally positive systems. Unlike the standard KYP lemma, the parametrized Lyapunov function $V(t) = x(t)^* Px(t)$ can be assumed diagonal without introducing conservatism. Most notably, this allows us to formulate a structured static state feedback $H_\infty$ control problem as an LMI problem. The newly uncovered structured static state feedback $H_\infty$ control is compared to the existing technique of structured static state feedback $L_\infty$-induced gain minimization control in the simulation study of the traffic control problem.

There are many open problems remaining in the control of positive systems.

- (Static vs. Dynamic controllers) It is known that the optimal dynamic state feedback controller can be found in the class of static state feedback. However it is not known whether this remains true if the controller is required to achieve internal positivity. To answer this question, more study is needed as to the class of dynamic controllers that attains internal positivity with respect to the plant state *only*, while allowing the controller state to be arbitrary. An immediate question is how to characterize such a class of dynamical controllers. To the best of our knowledge, this is not known. Structured dynamic $H_\infty$ controller synthesis achieving internal positivity is also unknown.
- (Linear vs. Nonlinear controllers) Although we have presented optimal structured controller syntheses in this chapter, this does not mean that the optimal structured static state feedback is necessarily linear. More study is needed to answer when the linear controller is optimal when a controller structure and the closed loop positivity is required.
- ($H_2$ and other criteria) When decentralized control design under the different performance criterion such as $H_2$ is considered, it is not clear what subclass of linear systems allows convex formulation of the problem. It is possible to formulate the (centralized) $H_2$ control problems as LMI problems. However, unlike the $H_\infty$ case we saw in this chapter, internal positivity of the system does not allow us to assume the Lyapunov-like parameter to be diagonal. Thus, internal positivity does not allow a convex formulation of the structured static state feedback control design. Hence, it remains an interesting question to ask what subclass of linear systems will yield a convex formulation of the structured control design.

# Chapter 5

# Conclusions and Future Works

## 5.1 Contribution of this thesis

This thesis considered three main themes: symmetric formulation of the KYP lemma, DC-dominant property of cone-preserving systems, and distributed control of positive systems. Technical contributions of this thesis are summarized as follows:

- The KYP lemma is presented as a systematic way to convert well-posedness problem into an efficiently verifiable convex programming. In the robust stability analysis, the proposed form of the KYP lemma treats "frequency" and "system uncertainties" equally, and helps to show connections between seemingly different problems in the literature.
- The notion of mutual losslessness is introduced to characterize a set of convex cones in the space of Hermitian matrices that yield a lossless S-procedure. Although no systematic method is known to check the mutual losslessness of a given set of Hermitian forms, this abstract notion explains the source of conservatism of various robustness analysis method including the KYP lemma.
- Cone-preserving transfer functions are considered, and this class of transfer function is shown to have the DC-dominant property. This is a valuable observation in that it eliminates a need of "frequency sweep" often required for robust stability and performance analysis.
- Using the DC-dominant property, the delay-independent stability is shown for the class of cone-preserving dynamical systems. In particular, this result is applied to give an alternative proof of the delay-independent mean-square stability of a multi-dimensional geometric Brownian motion.

- The "Diagonal" KYP lemma is shown to hold for internally positive systems. This result is particularly useful for a convex formulation of structured static state feedback control design problems.

## 5.2 Future Works

In this thesis, we have studied robustness analysis using the well-posedness model. The symmetric formulation of the KYP lemma (Theorem 1) is regarded as a systematic technique to relax the well-posedness condition to an efficiently verifiable matrix inequality condition. In this thesis, we have assumed that the space $\mathcal{R}(\mathbf{\Theta})$ of uncertain parameters given by

$$\mathcal{R}(\mathbf{\Theta}) := \left\{ S \in \mathbb{C}^{n\times n} : \begin{bmatrix} I \\ S \end{bmatrix}^* \Theta \begin{bmatrix} I \\ S \end{bmatrix} \geq 0 \quad \forall \Theta \in \mathbf{\Theta} \right\}$$

and this expression is used to characterize both regions of frequency variables and system uncertainties.

Some LMI techniques reported in the literature can be viewed as a generalization of the KYP lemma to the "higher order polynomials". In these techniques, the parameter space is given by

$$\mathcal{R}^{(m)}(\mathbf{\Theta}) := \left\{ S \in \mathbb{C}^{n\times n} : \begin{bmatrix} I \\ S \\ \vdots \\ S^{m-1} \end{bmatrix}^* \Theta \begin{bmatrix} I \\ S \\ \vdots \\ S^{m-1} \end{bmatrix} \geq 0 \quad \forall \Theta \in \mathbf{\Theta} \right\} \tag{5.1}$$

in an effort to improve computational performance or to reduce conservatism that is inevitable when the KYP lemma (Theorem 1) is used. In [85, 86], a frequency domain characterized similarly to (5.1) is referred to as the *generalized frequency*, and is used to reduce the size of the SDP to analyze the stability of interconnected identical subsystems. In [87], a use of Lyapunov functions taking higher order derivatives of the state is proposed for a less conservative robust stability analysis. In this viewpoint, $\Theta$ in (5.1) is understood to parameterize such Lyapunov functions. In [88], a similar expression to (5.1) is used to express parameter dependent Lyapunov functions (in contrast to the simultaneous Lyapunov functions for the quadratic stability), and a hierarchy of LMIs is proposed to analyze the stability of a linear dynamical system with multiple scalar uncertain parameters. Other higher order LMI techniques can be found in [89] and references therein. In the future, theoretical connections among these techniques, the formulation of the KYP lemma discussed in this thesis, and voluminous results algebraic geometry should be made transparent, so that flexible and efficient robust analysis method becomes available.

Although the considered approach provide a unified view on the robustness *analysis*, it it not clear how this observation can be effectively used in the control *synthesis* problems. This will be a future work as well.

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